J. Fluid Mech. (1990), *vol.* 216, *pp.* 133–160
Printed in Great Britain

Diffraction of sea waves by a slender body. Part 2. Water of finite depth

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(Received 26 October 1984 and in revised form 11 April 1989)

A uniformly valid theory (all wavelengths and angles of incidence) for the diffraction of sea waves by a slender body, correct to second order in the slenderness parameter, has been derived for the shallow-water limit. This theory is now extended to the finite water depth case, with the same results and accuracy.

1. Introduction

In an earlier paper (Aranha & Sugaya 1987, herein-after referred to as I) the diffraction of sea waves by a slender body was considered in the shallow-water limit. An asymptotic theory was then derived, uniformly valid for all frequencies and angles of incidence and correct to second order in the slenderness parameter. The purpose of the present work is to extend that theory to an arbitrary, although finite, water depth. The case where the depth is infinite will not be treated here and, as in paper I, the mean forward velocity is assumed to be zero.

Newman (1978) developed the leading-order term of this theory for the radiation problem in infinite depth and Sclavounos (1981) extended that work to the diffraction problem. More recently, Børresen & Faltinsen (1985) generalized the Newman–Sclavounos approach to finite depth. In all these works the apparent error factor, in the diffraction problem, is of the form $[1+O(\epsilon)]$, where ϵ is the slenderness parameter, and furthermore the kernel of the slender-body integral equation is relatively complicated, especially in the finite-depth case.

In the present work, the theory is formally derived up to second order in the slenderness parameter ϵ. In order to do so one must introduce, in a very precise way, how the error factor of the asymptotic theory is measured. As explained in I, several reasons have pushed us to measure this asymptotic error in the space of the generalized functions. This point is reviewed, and further discussed, in §6 of the present work, but one result must be stressed here: with this measure it is possible to show, for a class of slender geometries, that the leading-order solution already has an error factor of the form $[1+O(\epsilon^2)]$, even in the diffraction problem. This result is confirmed by some numerical experiments; see §7.

Besides this conceptual difference, related to the measurement of the error factor, the present work also introduces some simplifications from a more practical point of view. While Børresen & Faltinsen (1985) approach the finite-depth result from the Newman–Sclavounos infinite-depth theory, in the present work the finite-depth result is obtained from a different limit, namely from the shallow-water approximation, developed in I. As a consequence, the kernel of the slender-body integral equation is very simple here – it is just Hankel's function of the first kind – and at

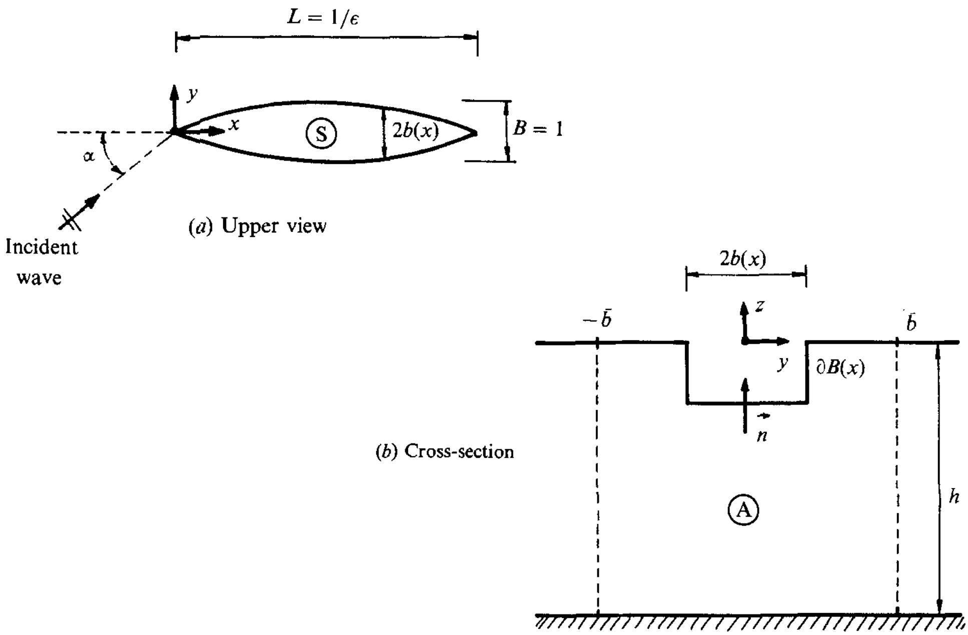

FIGURE 1. The geometric definitions of the problem under consideration. (a) Upper view, (b) cross-section.

the end of §5 the discrepancy between the present kernel and the one introduced in the Newman–Sclavounos theory is discussed from a more physical point of view.

We shall give here an overall view of the problem under consideration. The geometric definitions are given in figure 1, where α is the incidence angle; h the water depth; $\bar{b}$ the geometric parameter; A the finite fluid region $|y| \leqslant \bar{b}$; $\boldsymbol{n}$ the normal to the cross-section; $\boldsymbol{N} = \boldsymbol{n} + N_x \boldsymbol{i}$ the normal to S; S the body surface; and $\partial B(x)$ the contour line, section x.

The geometric parameter $\bar{b}$ must satisfy the inequality $\bar{b} \geqslant \frac{1}{2}B$, but it is otherwise arbitrary. Its introduction, together with the finite fluid region A, will be needed in §2, where the cross-section problem is analysed. In the normalization used here, $B = 1$ is the maximum beam and $L = 1/\epsilon$ the body length. Obviously, $\epsilon = B/L$ is the small slenderness parameter.

If K_0 is the wavenumber, related to the wave frequency ω by the dispersion relation,

$$\frac{\omega^2}{g} = K_0 \tanh K_0 h, \tag{1.1}$$

then the oblique incident wave has a factor $\exp(\mathrm{i}K_0 x \cos\alpha)$, rapidly oscillatory in the longitudinal direction. Following Newman (1978) and Sclavounos (1981), this rapidly oscillating term can be factored out, and the following definitions can be introduced:

$$\left.\begin{array}{ll} \text{incident wave} & \Phi_{\mathrm{I}}(y,z)\,\mathrm{e}^{\mathrm{i}K_0 x\cos\alpha}, \\ \text{scattered wave} & \Phi(x,y,z)\,\mathrm{e}^{\mathrm{i}K_0 x\cos\alpha}, \\ \text{total wave} & \Phi_{\mathrm{T}}(x,y,z)\,\mathrm{e}^{\mathrm{i}K_0 x\cos\alpha}. \end{array}\right\} \tag{1.2a}$$

A proper definition for the incident wave will be given in §2 and the total wave is defined by

$$\Phi_{\mathrm{T}}(x,y,z) = \Phi_{\mathrm{I}}(y,z)+\Phi(x,y,z). \tag{1.2b}$$

The scattered potential $\Phi(x,y,z)$ satisfies the field equation and boundary conditions

$$\left.\begin{aligned} &\nabla^2\Phi-(K_0\cos\alpha)^2\Phi+2\mathrm{i}K_0\cos\alpha\frac{\partial\Phi}{\partial x}+\frac{\partial^2\Phi}{\partial x^2}=0,\\ &\frac{\partial\Phi}{\partial z}=\frac{\omega^2}{g}\Phi;\quad z=0,\\ &\frac{\partial\Phi}{\partial z}=0;\quad z=-h,\end{aligned}\right\} \tag{1.3a}$$

together with the boundary condition at the body surface,

$$\nabla\Phi\cdot\boldsymbol{n}+N_x(\mathrm{i}K_0\cos\alpha\Phi+\partial\Phi/\partial x)\,|_{(x,y,z)\in S}=-\nabla\phi_{\mathrm{I}}\cdot\boldsymbol{n}-\mathrm{i}N_xK_0\cos\alpha\phi_{\mathrm{I}}\,|_{(x,y,z)\in S}, \tag{1.3b}$$

and the radiation condition

$$\Phi(r,\theta,z)\,\mathrm{e}^{\mathrm{i}K_0x\cos\alpha}\sim A(\theta)\frac{\mathrm{e}^{\mathrm{i}K_0r}}{(K_0r)^{\frac{1}{2}}}\cosh K_0(z+h);\quad r=(x^2+y^2)^{\frac{1}{2}}\to\infty. \tag{1.3c}$$

In the above expressions, and throughout this work, the time factor $\exp(-\mathrm{i}\omega t)$ is assumed and the following notation is used:

$$\nabla^2=\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\quad \nabla=\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}. \tag{1.4}$$

The method of matched asymptotic expansions, on which the slender-body theory is based, distinguishes two regions: one close to the body called the 'inner region', where $|y|/B \leqslant O(1)$, and the other far from it called the 'outer region', where $|y|/L \geqslant O(1)$. In the inner region the body is seen as if it were infinitely long, but the far-field radiation condition is excluded. It follows that the inner solution can be described by a cross-section problem (body appears infinitely long), although it cannot be completely determined (radiation condition missing). In the outer region the body is seen as if it were a line emitting waves, but the near-field body boundary condition excluded. The outer solution can be described, then, by an unknown distribution of wave singularities over the segment of length L. The indeterminacy of the inner and outer solutions is resolved by matching them in an 'overlap' region.

The solution of the cross-section problem is an essential step for the derivation of the inner solution. Since, however, there is some controversy concerning the head-sea limit of the cross-section solution, this point is analysed in some detail in §2 of the present work. In §3 the inner solution and its outer expansion are elaborated, and in §4 the outer solution and its inner expansion are derived. Both solutions are correct to second order in the slenderness parameter and are formally written in terms of some unknown functions, expressing the indeterminacy of each one. These functions are determined in §5, when the outer expansion of the inner solution is matched with the inner expansion of the outer solution. In the light of the error measure introduced here, and of some related results derived in I, the error factor of the second-order term is analysed in §6. It is shown, then, that for a useful class of slender geometries – those with an almost uniform cross-section – the leading-order solution already has an error factor of the form $1+O(\epsilon^2)$. In §7 the main conclusions of the present work are confirmed by some numerical experiments.

2. Cross-section solution

One of the reasons why the shallow-water limit was studied first is the simplicity of its cross-section problem. In this case the solution can be analytically determined and some relevant properties can be easily derived; see §2.1 in paper I. The purpose of the present section is to show that equivalent results can be derived for the arbitrary water depth case, but the formulation is much more complex and only the main features of this problem will be discussed. Mathematical details can be found in Aranha (1984), although the proofs of some more pertinent results are sketched in the Appendix of the present work.

Let $\{\omega; K_0\}$ be the wave frequency and wavenumber, related by the dispersion relation (1.1), and $f_0(z)$ be the function

$$\left.\begin{aligned} f_0(z) &= F_0 \cosh K_0(z+h), \\ F_0^2 &= \frac{1}{h}\frac{4K_0 h}{2K_0 h+\sinh 2K_0 h}; \quad \int_{-h}^{0} f_0^2(z)\,\mathrm{d}z = 1. \end{aligned}\right\} \tag{2.1}$$

The incident wave can be written as

$$\phi_{\mathrm{I}}(y,z) = A_{\mathrm{I}} f_0(z)\, \mathrm{e}^{\mathrm{i}K_0 y \sin\alpha}, \tag{2.2}$$

where $A_{\mathrm{I}} = \mathrm{i}(gA)/(F_0\, \omega \cosh K_0 h)$ and A is the wave amplitude. In the following it will be assumed that $A_{\mathrm{I}} = 1$. In the cross-section problem, the total, incident and scattered potentials will be designated by the lower-case letters $\{\phi_{\mathrm{T}}(y,z); \phi_{\mathrm{I}}(y,z); \phi(y,z)\}$, respectively, and the scattered potential must satisfy the equations

$$\left.\begin{aligned} &\nabla^2\phi-(K_0\cos\alpha)^2\phi = 0, \\ &\frac{\partial\phi}{\partial z} = \frac{\omega^2}{g}\phi; \quad z = 0, \\ &\frac{\partial\phi}{\partial z} = 0; \quad z = -h, \end{aligned}\right\} \tag{2.3}$$

together with the boundary condition on the cross-section contour line ∂B,

$$\nabla\phi \cdot \boldsymbol{n}|_{\partial B} = -(\nabla\phi_{\mathrm{I}} \cdot \boldsymbol{n})|_{\partial B}, \tag{2.4}$$

and the radiation condition

$$\phi(y,z) \sim \begin{Bmatrix} T(\alpha) \\ R(\alpha) \end{Bmatrix} f_0(z)\, \mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}; \quad y \to \pm\infty. \tag{2.5}$$

In (2.5), $R(\alpha)$ and $T(\alpha)+1$ are, respectively, the reflection and transmission coefficients. For a head sea ($\alpha = 0$) there is no physically clear radiation condition, but this point will be discussed later on in this section.

For $|y| \geqslant \bar{b} \geqslant b$, see figure 1, the fluid region is a rectangular strip and the solution of (2.3), (2.4) can be obtained by separation of variables. If $\phi^{\pm}(y,z)$ designate the solutions in the regions $y \gtrless \pm\bar{b}$, respectively, and $\phi(y,z)$ is the solution in the finite fluid region $A(|y| \leqslant \bar{b})$, the necessary and sufficient conditions for one to be the analytic continuation of the other are

$$\left.\begin{aligned} \phi(\pm\bar{b}, z) &= \phi^{\pm}(\pm\bar{b}, z), \\ \frac{\partial\phi}{\partial y}(\pm\bar{b}, z) &= \frac{\partial\phi^{\pm}}{\partial y}(\pm\bar{b}, z). \end{aligned}\right\} \tag{2.6}$$

Separation of variables, in the regions $y \lessgtr \pm\bar{b}$, leads naturally to a Sturm–Liouville problem and to a complete infinite set of orthonormal functions $\{f_n(z); n = 0, 1, 2, \ldots\}$, where $f_0(z)$ is given in (2.1) and, for $n \geqslant 1$,

$$\left.\begin{aligned} f_n(z) &= F_n \cos K_n(z+h); \quad \frac{\omega^2}{g} = -K_n \tan K_n h, \\ F_n^2 &= \frac{1}{h}\frac{4K_n h}{2K_n h - \sin 2K_n h}; \quad \int_{-h}^{0} f_n^2(z)\,\mathrm{d}z = 1. \end{aligned}\right\} \tag{2.7}$$

Since this set is orthonormal

$$\left(\int_{-h}^{0} f_n f_m \,\mathrm{d}z = \delta_{nm}\right)$$

and complete, the function $\phi(\pm\bar{b}, z)$ can be expressed as

$$\left.\begin{aligned} \phi(\pm\bar{b}, z) &= A_0^{\pm} f_0(z) + \sum_{n=1}^{\infty} L_n^{\pm}(\phi) f_n(z), \\ A_0^{\pm} &= L_0^{\pm}(\phi), \end{aligned}\right\} \tag{2.8}$$

where the linear functionals $L_n(\cdot)$ are defined by

$$L_n^{\pm}(\Psi) = \int_{-h}^{0} \Psi(\pm\bar{b}, z) f_n(z)\,\mathrm{d}z, \tag{2.9}$$

with $\Psi(y, z)$ being a function defined in A. It is an easy task to check that the solutions of (2.3), (2.5) in the regions $y \gtrless \pm\bar{b}$ that satisfy the continuity requirement (2.6) are given by

$$\left.\begin{aligned} \phi^{\pm}(y, z) &= A_0^{\pm} f_0(z)\,\mathrm{e}^{\mathrm{i}K_0(|y|-\bar{b})\sin\alpha} + \sum_{n=1}^{\infty} L_n^{\pm}(\phi) f_n(z)\,\mathrm{e}^{-\lambda_n(|y|-\bar{b})}, \\ \lambda_n(\alpha) &= [K_n^2 + (K_0\cos\alpha)^2]^{\frac{1}{2}}. \end{aligned}\right\} \tag{2.10}$$

Comparing (2.10) with (2.5) one obtains

$$\begin{Bmatrix} T(\alpha) \\ R(\alpha) \end{Bmatrix} = \begin{Bmatrix} A_0^{+}(\alpha) \\ A_0^{-}(\alpha) \end{Bmatrix} \mathrm{e}^{-\mathrm{i}K_0\bar{b}\sin\alpha}. \tag{2.11}$$

Enforcing continuity of velocity in $y = \pm\bar{b}$, see (2.6), the following expression can be derived, with the help of (2.10):

$$\frac{\partial\phi}{\partial y}(\pm\bar{b}, z) = \pm\mathrm{i}K_0 \sin\alpha A_0^{\pm} f_0(z) \mp \sum_{n=1}^{\infty} \lambda_n(\alpha) L_n^{\pm}(\phi) f_n(z). \tag{2.12}$$

The function $\phi(y, z)$ is a solution of (2.3), (2.4) in the finite fluid region A and it is subjected to (2.12) on the boundary lines $y = \pm\bar{b}$ of A. If the field equation in (2.3) is multiplied by $\Psi(y,z)$, integrated by parts in A and the boundary conditions in (2.3), (2.4) and (2.12) are used, the following identity is obtained:

$$G(\phi;\Psi)+\mathrm{i}K_0\sin\alpha[A_0^+L_0^+(\Psi)+A_0^-L_0^-(\Psi)] = K_0 V_0(\Psi), \tag{2.13}$$

where

$$G(\phi;\Psi) = \iint_A \nabla\phi\nabla\Psi\,\mathrm{d}A+(K_0\cos\alpha)^2\iint_A \phi\Psi\,\mathrm{d}A-\frac{\omega^2}{g}\int_F \phi(y,0)\,\Psi(y,0)\,\mathrm{d}y$$
$$+\sum_{n=1}^{\infty}\lambda_n(\alpha)\,[L_n^+(\phi)\,L_n^+(\Psi)+L_n^-(\phi)\,L_n^-(\Psi)] \tag{2.14}$$

and

$$V_0(\Psi) = -\frac{1}{K_0}\int_{\partial B}(\nabla\phi_{\mathrm{I}}\cdot\boldsymbol{n})\,\Psi\,\mathrm{d}\,\partial B. \tag{2.15}$$

In (2.14), F is the free surface of the finite fluid region A and (2.13) is well defined in the Hilbert space $W_2^{(1)}(A)$, of all $\Psi(y,z)$ whose gradient squared is integrable in A.

The diffraction in shallow water is described by the scattering matrix $\boldsymbol{A}$; see (2.13) in I. A similar formulation is possible here but, in order to do so, the undulatory and evanescent parts of $\phi(y,z)$ must be separated. With this motivation, the following Hilbert subspace is introduced:

$$W_{\mathrm{e}}(A) = \{\Psi_{\mathrm{e}}(y,z)\in W_2^{(1)}(A): L_0^{\pm}(\Psi_{\mathrm{e}}) = 0\}. \tag{2.16}$$

This subspace is called the 'evanescent' subspace of $W_2^{(1)}(A)$ since its elements, when extended to the regions $y \gtrless \pm\bar{b}$ by an expression similar to (2.10), have only evanescent terms. To complete the separation between undulatory and evanescent parts, the two auxiliary functions

$$q\pm(y,z) = \tfrac{1}{2}\left[1\pm\frac{y}{b}\right]f_0(z) \tag{2.17a}$$

are introduced, where obviously

$$L_0^{\pm}(q^{\pm}) = 1, \quad L_0^{\pm}(q^{\mp}) = 0. \tag{2.17b}$$

Elements of $W_2^{(1)}(A)$ can now be decomposed in the following way;

$$\left.\begin{aligned}\Psi(y,z) &= L_0^+(\Psi)\,q^+(y,z)+L_0^-(\Psi)\,q^-(y,z)+\Psi_{\mathrm{e}}(y,z); \quad \Psi_{\mathrm{e}}\in W_{\mathrm{e}}(A),\\ \phi(y,z) &= A_0^+\,q^+(y,z)+A_0^-\,q^-(y,z)+\phi_{\mathrm{e}}(y,z); \quad \phi_{\mathrm{e}}\in W_{\mathrm{e}}(A),\end{aligned}\right\} \tag{2.18}$$

where the relation $A_0^{\pm} = L_0^{\pm}(\phi)$, see (2.8), has been used. Placing (2.18) into (2.13) and introducing the linear functionals

$$V^{\pm}(\Psi) = -G(q^{\pm};\Psi), \tag{2.19}$$

it can be shown that $\phi(y,z)$ can be written as

$$\left.\begin{aligned}\phi(y,z) &= K_0\,\phi_{\mathrm{e}}^{(0)}(y,z)+A_0^+\,p^+(y,z)+A_0^-\,p^-(y,z),\\ p^{\pm}(y,z) &= q^{\pm}(y,z)+\phi_{\mathrm{e}}^{\pm}(y,z).\end{aligned}\right\} \tag{2.20}$$

The three unknown functions $\{\phi_{\mathrm{e}}^{(0)}(y,z);\phi_{\mathrm{e}}^{\pm}(y,z)\}$ are solutions of the following weak problem: to determine $\phi_{\mathrm{e}}(y,z) = \{\phi_{\mathrm{e}}^{(0)}(y,z);\phi_{\mathrm{e}}^{\pm}(y,z)\}\in W_{\mathrm{e}}(A)$ such that

$$G(\phi_{\mathrm{e}};\Psi_{\mathrm{e}}) = V(\Psi_{\mathrm{e}}), \quad \text{all} \quad \Psi_{\mathrm{e}}\in W_{\mathrm{e}}(A), \tag{2.21}$$

with $V(\cdot) = \{V_0(\cdot); V^{\pm}(\cdot)\}$; see (2.15) and (2.19). The two coefficients $A_0^{\pm}$ are solutions of the linear system

$$\boldsymbol{A}\cdot\begin{Bmatrix} A_0^+ \\ A_0 \end{Bmatrix} = \begin{Bmatrix} V_0^+ \\ V_0^- \end{Bmatrix}, \tag{2.22}$$

where

$$V_0^{\pm} = V_0(p^{\pm}), \tag{2.23a}$$

and the scattering matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{bmatrix} \dfrac{1}{K_0}G(p^+;p^+) - \mathrm{i}\sin\alpha & \dfrac{1}{K_0}G(p^+;p^-) \\ \dfrac{1}{K_0}G(p^+;p^-) & \dfrac{1}{K_0}G(p^-;p^-) - \mathrm{i}\sin\alpha \end{bmatrix}. \tag{2.23b}$$

From here on, the body will be assumed to be symmetric with respect to the plane $y=0$. In this case, the coefficients of the scattering matrix can be written in a more compact form, namely

$$\left.\begin{aligned} a_1 &= \frac{1}{K_0}G(p^+;p^+) = \frac{1}{K_0}G(p^-;p^-), \\ a_2 &= \frac{1}{K_0}G(p^+;p^-). \end{aligned}\right\} \tag{2.23c}$$

The result obtained here is analogous to the one derived in shallow water (see (2.13) in I), the only difference being that now the coefficients $\{a_1, a_2, V_0^{\pm}\}$ must be numerically determined.

Owing to the symmetry of the body, the potentials can be separated into symmetric and antisymmetric parts, designated in the following by the suffixes S and A, respectively. Observing that $(p^+(y,z)+p^-(y,z))$ is an even function of y and $(p^+(y,z)-p^-(y,z))$ is an odd function, then the symmetric and antisymmetric parts of the total potential ϕ_{T} can be written in the forms (see (1.2b) and (2.20))

$$\left.\begin{aligned} \phi_{\mathrm{T,S}}(y,z;\alpha) &= \phi_{\mathrm{I,S}}(y,z;\alpha) + K_0\,\phi^{(0)}_{\mathrm{e,S}}(y,z;\alpha) \\ &\quad + \tfrac{1}{2}(A_0^+(\alpha)+A_0^-(\alpha))\,[p^+(y,z;\alpha)+p^-(y,z;\alpha)], \\ \phi_{\mathrm{T,A}}(y,z;\alpha) &= \phi_{\mathrm{I,A}}(y,z;\alpha) + K_0\,\phi^{(0)}_{\mathrm{e,A}}(y,z;\alpha) \\ &\quad + \tfrac{1}{2}(A_0^+(\alpha)-A_0^-(\alpha))\,[p^+(y,z;\alpha)-p^-(y,z;\alpha)]. \end{aligned}\right\} \tag{2.24a}$$

In the above expression, the dependence on the incidence angle α is made explicit and $\{\phi_{\mathrm{I,S}};\phi_{\mathrm{I,A}}\}$ are the symmetric and anti-symmetric parts of the incident wave, namely

$$\left.\begin{aligned} \phi_{\mathrm{I,S}}(y,z;\alpha) &= \cos(K_0|y|\sin\alpha)\,f_0(z), \\ \phi_{\mathrm{I,A}}(y,z;\alpha) &= \mathrm{i}\sin(K_0|y|\sin\alpha)\,f_0(z). \end{aligned}\right\} \tag{2.24b}$$

The far-field behaviour of $\phi_{\mathrm{T}}(y,z;\alpha)$ can be obtained directly from either (2.24a) or (1.2b) and (2.11). To present this result in a more compact way, the following coefficients are introduced here:

$$\left.\begin{aligned} R_{\mathrm{S}}(\alpha) &= \tfrac{1}{2}(T(\alpha)+R(\alpha)), \\ R_{\mathrm{A}}(\alpha) &= \tfrac{1}{2}(T(\alpha)-R(\alpha)). \end{aligned}\right\} \tag{2.25a}$$

With (2.25*a*), the behaviour of $\phi_T(y,z;\alpha)$, when $|y| \to \infty$, is given by

$$\left.\begin{aligned}\phi_{T,S}(y,z;\alpha) &\sim [R_S(\alpha)\,e^{iK_0|y|\sin\alpha} + \cos(K_0|y|\sin\alpha)]\,f_0(z),\\ \phi_{T,A}(y,z;\alpha) &\sim [R_A(\alpha)\,e^{iK_0|y|\sin\alpha} + i\sin(K_0|y|\sin\alpha)]\,f_0(z).\end{aligned}\right\} \qquad (2.25b)$$

The results obtained so far are well known in the specialized literature. The reason for presenting them here, in such detail, is to give a theoretical background for the analysis of the second-order correction (see §3) and also to help the discussion of a controversial result, regarding the head-sea limit.

As would be expected, the cross-section solution depends continuously on the incidence angle α; see Aranha (1984). In this context, one must be careful to interpret the 'singular nature of the head-sea diffraction problem', established by Ursell (1968) and extensively quoted in the related literature (see, for example, Newman 1978 and Sclavounos 1981). In the following, this point will be discussed.

In the head-sea limit, the scattering matrix $\boldsymbol{A}$, given by (2.23*b*, *c*), is real and symmetric. It has, then, two real eigenvalues $\{\Lambda_1;\Lambda_2\}$, given by

$$\left.\begin{aligned}\Lambda_1 &= (a_1+a_2)_{\alpha=0},\\ \Lambda_2 &= (a_1-a_2)_{\alpha=0}.\end{aligned}\right\} \qquad (2.26)$$

The behaviour of the cross-section solution when $\alpha \to 0$ can be expressed with the help of the eigenvalues (2.26). Details can be found in the Appendix and only relevant results will be quoted here. Thus, the limit of the antisymmetric part of ϕ_T, when $\alpha \to 0$, is given by

$$\left.\begin{aligned}R_A(\alpha) &\sim i\sin\alpha\left(\frac{1}{\Lambda_2} - K_0\bar{b}\right) + O(\alpha^2),\\ \phi_{T,A}(y,z;\alpha) &\sim i\frac{\sin\alpha}{\Lambda_2}[p^+(y,z;0) - p^-(y,z;0)] + O(\alpha^2).\end{aligned}\right\} \qquad (2.27a)$$

As expected, the antisymmetric part tends to zero in the head-sea limit. The behaviour of the symmetric part is similar, since it is given by

$$\left.\begin{aligned}R_S(\alpha) &\sim -1 - i\sin\alpha\left(\frac{1}{\Lambda_1} - K_0\bar{b}\right) + O(\alpha^2),\\ \phi_{T,S}(y,z;\alpha) &\sim -i\frac{\sin\alpha}{\Lambda_1}[p^+(y,z;0) + p^-(y,z;0)],\end{aligned}\right\} \qquad (2.27b)$$

where, when $\alpha = 0$, the following identity also holds:

$$p^+(y,z;0) + p^-(y,z;0) = K_0\,\phi_e^{(0)}(y,z;0) + f_0(z). \qquad (2.27c)$$

In the shallow-water limit, the cross-section solution can be determined analytically and the same result, (2.27*b*), can be obtained: see (2.19), (2.20) in I. This fact, and also the discussion to follow, gives confidence in the validity of (2.27), even without the demonstration in the Appendix.

The cross-section solution is then *regular* in the limit $\alpha \to 0$. The origin of the misunderstanding seems to be related to the fact that $\phi_T(y,z;\alpha) \to 0$, when $\alpha \to 0$. This result could be anticipated from the following considerations: when $\alpha = 0$, $f_0(z)$ is a solution of (2.3) and the incident wave is given by $\phi_I = f_0(z)$ (see (2.2)). For a symmetric body in a head sea, the 'radiation condition' (2.5) can be written in the form $\phi \sim Tf_0(z)$ and, from the boundary condition on the body surface, one must have $\phi = -f_0(z)$. Thus, $\phi_T = \Phi_I + \phi \equiv 0$, as indicated by (2.27*b*).

If one observes that $yf_0(z)$ is also a solution of (2.3), one could specify the far-field behaviour

$$\phi(y,z) \sim [A_0 + K_0 U(|y| - \bar{b})] f_0(z); \quad A_0 = L_0(\phi), \tag{2.28}$$

instead of (2.5), to obtain a non-trivial solution ϕ_T. In this case, the head-sea solution is given exactly by (2.27*b*), with U in place of $-\mathrm{i}\sin\alpha$; see Appendix. Obviously, the solution, although non-trivial, cannot be determined, since the value of U is unknown.

Ursell (1968) observed that the two-dimensional Green's function is composed of two parts when $\alpha \to 0$: one that behaves like $1/\sin\alpha$, and the other that is regular in this limit. In his analysis, he discarded the first part and, considering only the regular part, he obtained solutions with the behaviour (2.28). The scattered wave, however, is represented by an integral over the body of the Green's function multiplied by the source density $\sigma(y,z;\alpha)$. Since the source density is proportional to the total potential, then $\sigma(y,z;\alpha) \sim O(\sin\alpha)$† when $\alpha \to 0$; see (2.27*b*). In this circumstance, the contribution of the regular part of the Green's function tends to zero, like the terms proportional to $-\mathrm{i}\sin\alpha$ in (2.7*b*), and the contribution of the 'irregular part', proportional to $1/\sin\alpha$, gives a finite limit for the scattered wave.

In the slender-body theory, the potential $\phi_{T,S}(y,z;\alpha)$ is multiplied by an unknown function $S(x;\alpha)$, which is determined in the matching process. It can be shown that $S(x;\alpha)$ behaves like $1/\sin\alpha$ when $\alpha \to 0$, and so the product $S(x;\alpha)\,\phi_{T,S}(y,z;\alpha)$ is regular and non-zero when $\alpha = 0$. This result, clearly, is a consequence of the regularity in α of both the slender-body theory and the cross-section problem.

In the short-wave limit ($K_0 \to \infty$), the full three-dimensional solution $\Phi_T(x,y,z;\alpha)$ should approach the cross-section solution $\phi_T(y,z;\alpha)$. This is the classical 'strip-theory approximation' that is thought to be valid only when $\alpha \neq 0$. Arguments to be given next show that this result is also correct in the head-sea limit.

In fact, it is a well-known result (see Faltinsen 1971; Haren & Mei 1981) that $S(x;0) \sim O((K_0 x)^{-\frac{1}{2}})$‡ when $K_0 x \gg 1$, where $S(x;0)$ is the slender-body multiplicative function in a head sea. This result indicates that $\Phi_T(x,y,z;0) \to 0$, when $K_0 \to \infty$ or, in short, the head sea is refracted away by a slender body; see Ursell (1977). Since $\phi_T(y,z;\alpha) \to 0$, when $\alpha \to 0$, the above result allows one to write

$$\lim_{K_0 \to \infty} \Phi_T(x,y,z;\alpha) = \phi_T(y,z;\alpha), \tag{2.29}$$

for *all* incidence angles α. Or, in short, within this interpretation, the strip-theory approximation is the short-wave limit of the full three-dimensional solution, for arbitrary α.

One could use Ursell's solution in a head sea, with the unknown parameter U, and match the inner and outer solutions in two steps: one when $\alpha > 0$; the other when $\alpha = 0$. In this case, it must be shown that the latter is the limit of the former when $\alpha \to 0$, since the slender-body solution should necessarily be continuous in α. This procedure was used in I but, although producing the correct result, it is somewhat

† Note, however, that Faltinsen (1971) used only the regular part of the Green's function and imposed (2.28) with $A_0 = K_0 U\bar{b}$. In this way he obtained a source density $\sigma_1(y,z) \neq \lim_{\alpha\to 0}\sigma(y,z;\alpha) = 0°$.

‡ For a particular class of geometries (small draught) and in a certain range of frequencies. $S(x;0)$ remains oscillatory when $X \to \infty$, with K_0 finite. If $K_0 \to \infty$, however, $S(x;0) \to 0$, and so (2.29) continues to be valid. See Aranha & Sugaya (1985).

artificial. In the present work, the cross-section solution $\phi_T(y,z;\alpha)$ will be used, for all α, and the head-sea solution will come out naturally. This is not only a more direct approach but also it makes explicit the regularity of the cross-section solution in the head-sea limit.

3. The inner solution and its outer expansion

In the inner region, the transverse lengthscale is $B = 1$ and the longitudinal lengthscale is $L = 1/\epsilon$. If $f(x,y,z)$ is any flow or geometric variable, the slenderness assumption leads to the following estimate for the order of magnitude of the gradient of $f(x,y,z)$ (see (1.3)):

$$\frac{\partial f}{\partial x} \sim O(\epsilon f), \quad \nabla f \sim O(f). \tag{3.1}$$

Observing that N_x is of order ϵ in a slender body then, disregarding terms of order ϵ^2 compared with 1, one obtains the following set of equations in the inner region:

$$\left.\begin{aligned} &\nabla^2\Phi - (K_0\cos\alpha)^2\Phi = -2\mathrm{i}K_0\cos\alpha\frac{\partial\Phi}{\partial x}, \\ &\nabla\phi\cdot\boldsymbol{n}|_{\partial B} = -(\nabla\phi_{\mathrm{I}}\cdot\boldsymbol{n})_{\partial B} - \mathrm{i}N_x K_0\cos\alpha(\phi_{\mathrm{I}}+\Phi)_{\partial B}, \end{aligned}\right\} \tag{3.2a}$$

together with the boundary conditions

$$\left.\begin{aligned} &\frac{\partial\Phi}{\partial z} = \frac{\omega^2}{g}\Phi; \quad z = 0, \\ &\frac{\partial\Phi}{\partial z} = 0; \quad z = -h. \end{aligned}\right\} \tag{3.2b}$$

In (3.2*a*), the terms proportional to $\{\partial\Phi/\partial x; N_x\}$ are of relative order ϵ and so, to leading order, (3.2) are coincident with the cross-section equations (2.3), (2.4). Obviously, the three-dimensional radiation condition (1.3) cannot be used in this context and so the solution of (3.2) cannot be completely determined. The inner solution will be designated by the suffix i and, correct to second order in the slenderness parameter, it is given by

$$\left.\begin{aligned} &\Phi_{\mathrm{i}}(x,y,z) = \Phi_{\mathrm{i}}^{(1)}(x,y,z) + \Phi_{\mathrm{i}}^{(2)}(x,y,z), \\ &\Phi_{\mathrm{i}}^{(2)}/\Phi_{\mathrm{i}}^{(1)} \sim O(\epsilon). \end{aligned}\right\} \tag{3.3}$$

In the nomenclature of the slender-body theory, $\{\Phi_{\mathrm{i}}^{(1)};\Phi_{\mathrm{i}}^{(2)}\}$ are called the leading-order and second-order inner solutions, respectively. Placing (3.3) into (3.2*a*), and separating terms of the same order in ϵ, one obtains two sets of 'cross-section equations' for $\Phi_{i}^{(1)}$ and $\Phi_{\mathrm{i}}^{(2)}$. The equations, and their respective solutions, will be analysed in the following.

3.1. *The leading-order solution*

Since the terms proportional to $\{\partial\Phi/\partial x; N_x\}$ are of relative order ϵ, the leading-order solution satisfies the boundary conditions (3.2*b*) and the following set of equations:

$$\left.\begin{aligned} &\nabla^2\Phi_{\mathrm{i}}^{(1)} - (K_0\cos\alpha)^2\Phi_{\mathrm{i}}^{(1)} = 0, \\ &\nabla\Phi_{\mathrm{i}}^{(1)}\cdot\boldsymbol{n}|_{\partial B} = -(\nabla\phi_{\mathrm{I}}\cdot\boldsymbol{n})|_{\partial B}. \end{aligned}\right\} \tag{3.4}$$

The potential $\Phi_{\mathrm{i}}^{(1)}$ is forced by the boundary condition on the body surface; see (3.4).

Since *two* boundary conditions are ignored (namely, the radiation conditions when $y \to \pm \infty$) the most general solution of (3.4) and (3.2*b*) can be written as a sum of a particular solution with *two* linearly independent homogeneous solutions.

The particular solution can be defined by the expression

$$\phi_{\mathrm{P}}^{(1)} = -\phi_{\mathrm{I}}, \tag{3.5a}$$

since $-\phi_{\mathrm{I}}$ satisfies all conditions in (3.2*b*) and (3.4). Observing that the symmetric and antisymmetric parts of the cross-section total wave, $\{\phi_{\mathrm{T,S}}(y,z;\alpha); \phi_{\mathrm{T,A}}(y,z;\alpha)\}$ respectively, are linearly independent functions that satisfy the homogeneous boundary condition on the body surface, the homogeneous solution of (3.2*b*), (3.4) can be written in the form

$$\phi_{\mathrm{H}}^{(1)} = S_1(x;\alpha)\,\phi_{\mathrm{T,S}}(y,z;\alpha) + (\operatorname{sgn} y)\,A_1(x;\alpha)\,\phi_{\mathrm{T,A}}(y,z;\alpha). \tag{3.5b}$$

In the context of the cross-section problem (3.2), (3.4), $\{S_1(x;\alpha); A_1(x;\alpha)\}$ are two arbitrary 'constants'. Obviously, these 'constants' should not be the same in different sections and this is the reason why both are functions of the x-section. If the body is not cylindrical, the cross-section geometry changes with x and so the total potential ϕ_{T} also does. To keep clear, however, that this is a two-dimensional solution, the x-dependence will not be made explicit in it. From (3.5*a*, *b*) it follows that

$$\begin{aligned}\Phi_{\mathrm{i}}^{(1)}(x,y,z;\alpha) = {} & [S_1(x;\alpha)\,\phi_{\mathrm{T,S}}(y,z;\alpha) - \cos(K_0|y|\sin\alpha)\,f_0(z)] \\ & + (\operatorname{sgn} y)\,[A_1(x;\alpha)\,\phi_{\mathrm{T,A}}(y,z;\alpha) - \mathrm{i}\sin(K_0|y|\sin\alpha)\,f_0(z)].\end{aligned} \tag{3.5c}$$

These simple expressions for the particular and homogeneous solutions have been proposed by Newman (1978) and, in the following, the outer expansion of the leading-order inner solution will be elaborated.

The matching between inner and outer solutions should be enforced in an overlap region, where $B \ll y$ or $1 \ll y/B \ll 1/\epsilon$. This region can then be characterized by the relation $|y|/\bar{b} = \epsilon^{-\nu}$, where $0 < \nu < 1$, and $\bar{b}$ is the half-beam of the widest section ($\bar{b} = \frac{1}{2}B$). Observing that $L_n^{\pm}(\phi_{\mathrm{T}}) = L_n^{\pm}(\phi)$, for $n \geqslant 1$, and introducing the definitions

$$\left.\begin{aligned} L_{n,\mathrm{S}}(\phi_{\mathrm{T}}) &= \tfrac{1}{2}(L_n^+(\phi) + L_n^-(\phi)), \\ L_{n,\mathrm{A}}(\phi_{\mathrm{T}}) &= \tfrac{1}{2}(L_n^+(\phi) - L_n^-(\phi)), \end{aligned}\right\} \tag{3.6}$$

the *evanescent* parts of $\phi_{\mathrm{T,S}}(y,z;\alpha)$ and $\phi_{\mathrm{T,A}}(y,z;\alpha)$, in the region $|y| > \bar{b}$, are given by (see (2.10))

$$\begin{Bmatrix} \phi_{\mathrm{T,S}}^{(\mathrm{e})}(y,z;\alpha) \\ \phi_{\mathrm{T,A}}^{(\mathrm{e})}(y,z;\alpha) \end{Bmatrix} = \begin{Bmatrix} \sum_{n=1}^{\infty} & L_{n,\mathrm{S}}(\phi_{\mathrm{T}}) \\ & L_{n,\mathrm{A}}(\phi_{\mathrm{T}}) \end{Bmatrix} f_n(z)\,\mathrm{e}^{-\lambda_n(|y|-b)}. \tag{3.7a}$$

Since $\lambda_n h > K_n h > \frac{1}{2}(2n-1)\pi$, where h is the water depth (see (2.7), (2.10)), then

$$\mathrm{e}^{-\lambda_n|y|} < \exp\left[-\frac{\pi\bar{b}}{2h}\frac{|y|}{\bar{b}}\right] = \exp\left[-\left(\frac{\pi\bar{b}}{2h}\right)\frac{1}{\epsilon^{\nu}}\right], \tag{3.7b}$$

in the overlap region $|y|/\bar{b} = \epsilon^{-\nu}$. The asymptotic theory is to be developed in the limit $\epsilon \to 0$ and both $\bar{b}$ and h are independent of ϵ. Expression (3.7*b*) shows then, that the *evanescent terms are exponentially small in the overlap region.*

The outer expansion of the inner solution is just the asymptotic behaviour of the inner solution in the overlap region. From the above result and (3.5c) it follows then (see also (2.25b)) that

$$\begin{aligned}\phi_{\mathrm{i}}^{(1)}(x,y,z;\alpha) \sim \{&[S_1(x,\alpha)(1+R_{\mathrm{S}}(\alpha))-1]\cos(K_0|y|\sin\alpha)\\ &+\mathrm{i}S_1(x;\alpha)R_{\mathrm{S}}(\alpha)\sin(K_0|y|\sin\alpha)\}f_0(z)\\ &+(\operatorname{sgn} y)\{A_1(x;\alpha)R_{\mathrm{A}}(\alpha)\cos(K_0|y|\sin\alpha)\\ &+\mathrm{i}[A_1(x;\alpha)(R_{\mathrm{A}}(\alpha)+1)-1]\sin(K_0|y|\sin\alpha)\}f_0(z).\end{aligned} \tag{3.8}$$

Observe, again, that $\{R_{\mathrm{S}}(\alpha);R_{\mathrm{A}}(\alpha)\}$ are also functions of x if the body is not cylindrical.

3.2. *The second-order solution*

When (4.3) is put into (4.2a), the resulting equations for $\Phi_{\mathrm{i}}^{(2)}$ are given by

$$\nabla^2\Phi_{\mathrm{i}}^{(2)}-(K_0\cos\alpha)^2\,\Phi_{\mathrm{i}}^{(2)} = -2\mathrm{i}K_0\cos\alpha\frac{\partial\Phi_{\mathrm{i}}^{(1)}}{\partial x}, \tag{3.9a}$$

$$\nabla\Phi_{\mathrm{i}}^{(2)}\cdot\boldsymbol{n}\,|_{\partial B} = -\mathrm{i}N_x K_0\cos\alpha(\phi_{\mathrm{I}}+\Phi_{\mathrm{i}}^{(1)})\,|_{\partial B}. \tag{3.9b}$$

The most general solution of (4.2b), (4.9) can again be written in the form

$$\begin{aligned}\Phi_{\mathrm{i}}^{(2)}(x,y,z;\alpha) = {}&[S_2(x;\alpha)\,\phi_{\mathrm{T,S}}(y,z;\alpha)+\phi_{\mathrm{P,S}}(y,z;\alpha)]\\ &+(\operatorname{sgn} y)[A_2(x;\alpha)\,\phi_{\mathrm{T,A}}(y,z;\alpha)+\phi_{\mathrm{P,A}}(y,z;\alpha)],\end{aligned} \tag{3.10}$$

where $\{S_2(x;\alpha);A_2(x;\alpha)\}$ are the two arbitrary cross-section 'constants' and $\phi_{\mathrm{P}} = \phi_{\mathrm{P,S}}+(\operatorname{sgn} y)\,\phi_{\mathrm{P,A}}$ is a particular solution of (3.2b) and (3.9). (Again, ϕ_{P} is a function of x; since, however, it is a cross-section solution this dependence is omitted.) In the following the derivation of a convenient ϕ_{P} will be elaborated.

As in §2, $\phi_{\mathrm{P}}^{\pm}$ will designate the particular solution in the regions $y \gtrless \pm\bar{b}$ and ϕ_{P} the particular solution in the finite fluid region A. If $\Gamma^{\pm}(y,z;\alpha)$ is a particular solution of (3.9a), with

$$\Gamma^{\pm}(\pm\bar{b},z;\alpha) \equiv 0, \tag{3.11a}$$

then, from the continuity condition

$$\phi_{\mathrm{P}}^{\pm}(\pm\bar{b},z;\alpha) = \phi_{\mathrm{P}}(\pm\bar{b},z;\alpha), \tag{3.11b}$$

one must have (see (2.9))

$$\phi_{\mathrm{P}}^{\pm}(y,z;\alpha) = \Gamma^{\pm}(y,z;\alpha)+L_0^{\pm}(\phi_{\mathrm{P}})\,\mathrm{e}^{\mathrm{i}K_0\sin\alpha(|y|-\bar{b})}f_0(z)+\sum_{n=1}^{\infty}L_n^{\pm}(\phi_{\mathrm{P}})\,\mathrm{e}^{-\lambda_n(|y|-\bar{b})}f_n(z). \tag{3.11c}$$

Since $\partial\Phi_{\mathrm{i}}^{(1)}/\partial x$ is expressed by means of a series in the regions $y \gtrless \pm\bar{b}$ the function $\Gamma^{\pm}(y,z;\alpha)$ can be determined analytically. In fact, if

$$\begin{aligned}C_{0,\mathrm{S}}(y,\alpha) = {}&-\mathrm{i}\frac{\cos\alpha}{\sin\alpha}|y|\sin(K_0|y|\sin\alpha)\frac{\mathrm{d}}{\mathrm{d}x}[S_1(x;\alpha)(1+R_{\mathrm{S}}(\alpha))-1]\\ &-\mathrm{i}\left(\frac{\cos\alpha}{K_0\sin^2\alpha}\sin(K_0|y|\sin\alpha)-\frac{\cos\alpha}{\sin\alpha}|y|\cos(K_0|y|\sin\alpha)\right)\frac{\mathrm{d}}{\mathrm{d}x}[iS_1(x;\alpha)R_{\mathrm{S}}(\alpha)],\end{aligned} \tag{3.11d}$$

and

$$\begin{aligned}C_{0,\mathrm{A}}(y,\alpha) = {}&-\mathrm{i}\frac{\cos\alpha}{\sin\alpha}|y|\sin(K_0|y|\sin\alpha)\frac{\mathrm{d}}{\mathrm{d}x}[A_1(x;\alpha)R_{\mathrm{A}}(\alpha)]\\ &-\frac{\cos\alpha}{\sin\alpha}|y|\cos(K_0|y|\sin\alpha)\frac{\mathrm{d}}{\mathrm{d}x}[A_1(x;\alpha)(R_{\mathrm{A}}(\alpha)+1)-1],\end{aligned} \tag{3.12b}$$

then $[C_{0,\mathrm{S}}(y;\alpha) \pm C_{0,\mathrm{A}}(y;\alpha)]\, f_0(z)$ is a solution of (3.9*a*) when $\Phi_{\mathrm{I}}^{(1)}$ is given by its asymptotic expression (3.8). Notice that the homogeneous term, proportional to sin $(K_0|y|\sin\alpha)$ in (3.12*a*), is needed to render $C_{0,\mathrm{S}}(y;\alpha)$ regular in the limit $\alpha \to 0$. (A homogeneous term is not needed for $C_{0,\mathrm{A}}(y;\alpha)$, since the antisymmetric part is known to be zero in a head sea.) Similar expressions can be obtained for the evanescent terms of $\phi_{\mathrm{I}}^{(1)}$. In fact, if

$$\left.\begin{aligned} C_{n,\mathrm{S}}(y;\alpha) &= -\mathrm{i}\frac{K_0\cos\alpha}{\lambda_n}|y|\,\mathrm{e}^{-\lambda_n(|y|-\bar{b})}\frac{\mathrm{d}}{\mathrm{d}x}[S_1(x;\alpha)\,L_{n,\mathrm{S}}(\phi_{\mathrm{T}})], \\ C_{n,\mathrm{A}}(y;\alpha) &= \mathrm{i}\frac{K_0\cos\alpha}{\lambda_n}|y|\,\mathrm{e}^{-\lambda_n(|y|-\bar{b})}\frac{\mathrm{d}}{\mathrm{d}x}[A_1(x;\alpha)\,L_{n,\mathrm{A}}(\phi_{\mathrm{T}})], \end{aligned}\right\} \tag{3.13a}$$

then $\Sigma\,[C_{n,\mathrm{S}}(y;\alpha) \pm C_{n,\mathrm{A}}(y,\alpha)]\, f_n(z)$ is a solution of (3.9*a*), when $\Phi_{\mathrm{I}}^{(1)}$ is given by (3.5*c*) and (3.7*a*). Introducing the definitions

$$C_n^{\pm}(y;\alpha) = C_{n,\mathrm{S}}(y;\alpha) \pm C_{n,\mathrm{A}}(y;\alpha); \quad n = 0, 1, \ldots, \tag{3.13b}$$

it is now an easy task to verify that

$$\begin{aligned} \Gamma^{\pm}(y,z;\alpha) = {} & [C_0^{\pm}(y,\alpha) - C_0^{\pm}(\bar{b};\alpha)\,\mathrm{e}^{\mathrm{i}K_0\sin\alpha(|y|-\bar{b})}]\, f_0(z) \\ & + \sum_{n=1}^{\infty} [C_n^{+}(y;\alpha) - C_n^{\pm}(\bar{b};\alpha)\,\mathrm{e}^{-\lambda_n(|y|-\bar{b})}]\, f_n(z), \end{aligned} \tag{3.13c}$$

is a solution of (3.2*b*), (3.9*a*), (3.11*a*), in the regions $y \gtrless \pm\bar{b}$. Observing that

$$\frac{\partial\Gamma^{\pm}}{\partial y}(\pm\bar{b},z;\alpha) = \sum_{n=0}^{\infty} \gamma_n^{\pm} f_n(z), \tag{3.14a}$$

with

$$\left.\begin{aligned} \gamma_0^{\pm} &= \pm\left[\frac{\mathrm{d}C_{0,\mathrm{S}}}{\mathrm{d}y}(\bar{b};\alpha) - \mathrm{i}K_0\sin\alpha C_{0,\mathrm{S}}(\bar{b};\alpha)\right] + \left[\frac{\mathrm{d}C_{0,\mathrm{A}}}{\mathrm{d}y}(\bar{b};\alpha) - \mathrm{i}K_0\sin\alpha C_{0,\mathrm{A}}(\bar{b};\alpha)\right], \\ \gamma_n^{\pm} &= \pm\left[\frac{\mathrm{d}C_{n,\mathrm{S}}}{\mathrm{d}y}(\bar{b};\alpha) + \lambda_n C_{n,\mathrm{S}}(\bar{b};\alpha)\right] + \left[\frac{\mathrm{d}C_{n,\mathrm{A}}}{\mathrm{d}y}(\bar{b};\alpha) + \lambda_n C_{n,\mathrm{A}}(\bar{b};\alpha)\right], \end{aligned}\right\} \tag{3.14b}$$

then, imposing continuity of velocity on $y = \pm\bar{b}$, one obtains

$$\frac{\partial\phi_{\mathrm{P}}}{\partial y}(\pm\bar{b},z;\alpha) = \pm\mathrm{i}K_0\sin\alpha L_0^{\pm}(\phi_{\mathrm{P}})\, f_0(z) \mp \sum_{n=1}^{\infty} \lambda_n(\alpha)\, L_n^{\pm}(\phi_{\mathrm{P}})\, f_n(z) + \sum_{n=1}^{\infty} \gamma_n^{\pm} f_n(z). \tag{3.14c}$$

In the finite fluid region A, $\phi_{\mathrm{P}}(y,z;\alpha)$ satisfies (3.2*b*), (3.9) and (3.14) on its boundary $y = \pm\bar{b}$. Multiplying the field equation (3.9*a*) by a function $\Psi(y,z)$ and integrating by parts in A, the following identity can be derived:

$$G(\phi_{\mathrm{P}};\Psi) + \mathrm{i}K_0\sin\alpha[L_0^{+}(\phi_{\mathrm{P}})L_0^{+}(\Psi) + L_0^{-}(\phi_{\mathrm{P}})L_0^{-}(\Psi)] = K_0\, V_1(\Psi), \tag{3.15a}$$

where $G(.;.)$ is given by (2.14) and

$$\begin{aligned} K_0\, V_1(\Psi) = {} & \sum_{n=0}^{\infty} [\gamma_n^{+} L_n^{+}(\Psi) - \gamma_n^{-} L_n^{-}(\Psi)] + 2\mathrm{i}K_0\cos\alpha \iint_A \frac{\partial\Phi_{\mathrm{I}}^{(1)}}{\partial x}\Psi\,\mathrm{d}A \\ & - \mathrm{i}N_x K_0\cos\alpha \int_{\partial B} (\phi_{\mathrm{I}} + \Phi_{\mathrm{I}}^{(1)})\,\Psi\,\mathrm{d}\,\partial B. \end{aligned} \tag{3.15b}$$

The formal similarity between (2.13) and (3.15*a*) allows one to write directly the final expression for $\phi_P(y,z;\alpha)$. In fact, if $\phi_e^{(1)}(y,z;\alpha)$ is the solution of the weak equation (see (2.21))

$$G(\phi_e^{(1)};\Psi_e) = V_1(\Psi_e), \quad \text{all} \quad \Psi_e \in W_e(A), \tag{3.16a}$$

and (see (2.23*a*))

$$V_1^{\pm} = V_1(p\pm), \tag{3.16b}$$

then (see (2.20))

$$\phi_P(y,z;\alpha) = K_0\,\phi_e^{(1)}(y,z;\alpha) + L_0^+(\phi_P)\,p^+(y,z) + L_0^-(\phi_P)\,p^-(y,z), \tag{3.16c}$$

where $L_0^{\pm}(\phi_P)$ are solutions of the linear system (see (2.22))

$$\boldsymbol{A}\begin{Bmatrix} L_0^+(\phi_P) \\ L_0^-(\phi_P)\end{Bmatrix} = \begin{Bmatrix} V_1^+ \\ V_1^- \end{Bmatrix}. \tag{3.16d}$$

Obviously, the particular solution $\phi_P(y,z;\alpha)$ depends on the yet unknown constants $\{S_1(x;\alpha);A_1(x;\alpha)\}$ of the leading-order solution. Once they are determined, $\phi_P(y,z;\alpha)$ can be determined from the leading-order matching with the outer solution by means of (3.16). If

$$\left.\begin{aligned} D_S(\alpha) &= [\tfrac{1}{2}(L_0^+(\phi_P)+L_0^-(\phi_P)) - C_{0,S}(\bar{b};\alpha)]\,e^{-iK_0\bar{b}\sin\alpha}, \\ D_A(\alpha) &= [\tfrac{1}{2}(L_0^+(\phi_P)+L_0^-(\phi_P)) - C_{0,A}(\bar{b};\alpha)]\,e^{-iK_0\bar{b}\sin\alpha}, \end{aligned}\right\} \tag{3.17a}$$

the outer expansion of the second-order inner solution is given (see (2.25*b*), (3.10), (3.11*c*), (3.13*b, c*)) by

$$\begin{aligned} \Phi_1^{(2)}(x,y,z;\alpha) \sim{} & \{[S_2(x;\alpha)(1+R_S(\alpha)) + D_S(\alpha))]\cos(K_0|y|\sin\alpha) \\ & + i(S_2(x;\alpha)R_S(\alpha) + D_S(\alpha))\sin(K_0|y|\sin\alpha)\}\,f_0(z) \\ & + (\operatorname{sgn} y)\{(A_2(x;\alpha)R_A(\alpha) + D_A(\alpha))\cos(K_0|y|\sin\alpha) \\ & + i[A_2(x;\alpha)(R_A(\alpha)+1) + D_A(\alpha)]\sin(K_0|y|\sin\alpha)\}\,f_0(z) \\ & + [C_{0,S}(y,\alpha) + (\operatorname{sgn} y)\,C_{0,A}(y;\alpha)]\,f_0(z). \end{aligned} \tag{3.17b}$$

Again, the contribution of the evanescent terms is exponentially small in the overlap region.

4. The outer solution and its inner expansion

In the outer region ($|y|/L \geqslant O(1)$), the body shrinks to a line and the boundary condition (1.3*b*) is lost. The solution in this region, named the outer solution, can then be represented by a distribution of singularities (poles, dipoles, etc.) along the line $0 \leqslant x \leqslant L$, since each singularity satisfies (1.3*a, c*). (In reality, the singularities should be distributed along $0^+ \leqslant x \leqslant L^-$ to avoid the end points $x=0; x=L$; see (6.2*b*) and (7.34*a*) in I. This observation seems to be consistent, within an error ϵ^2, with Geer's (1975) work, where the outer solution is defined on $\delta_1 \leqslant x \leqslant L-\delta_2$ with δ_1, δ_2 of order ϵ^2.) The linear densities of these singularities are unknown, reminiscent of the unknown body boundary condition. They will be determined by matching with the inner solution, which contains just two unknowns $\{S(x;\alpha);A(x;\alpha)\}$; see §3. So, there must exist an even singularity (pole), with density $Q(x;\alpha)$, and another odd one (dipole), with density $M(x;\alpha)$. The first is associated with $S(x;\alpha)$ and the second with $A(x;\alpha)$.

Expanding the densities in the small parameter ϵ, one may write

$$\left.\begin{aligned} Q(x;\alpha) &= Q_1(x;\alpha)+Q_2(x;\alpha), \\ M(x;\alpha) &= M_1(x;\alpha)+M_2(x;\alpha) \\ \{Q_2/Q_1;M_2/M_1\} &\sim O(\epsilon). \end{aligned}\right\} \tag{4.1a}$$

The outer solution, designated by the suffix o, can be written then in the following way:

$$\Phi_{\mathrm{o}}(x,y,z;\alpha) = \int_0^L \mathrm{d}\xi\left[Q(\xi;\alpha)+\frac{1}{K_0}M(\xi;\alpha)\frac{\partial}{\partial y}\right]\bar{G}(x-\xi,y,z), \tag{4.1b}$$

where $\bar{G}(.,.,.)$ is a Green's function multiplied by the constant factor $1/f_0(0)$, or

$$\left.\begin{aligned} \bar{G}(x,y,z;\alpha) &= \left[-\frac{\mathrm{i}}{4}H_0^{(1)}(K_0(x^2+y^2)^{\frac{1}{2}})\,\mathrm{e}^{-\mathrm{i}K_0x\cos\alpha}\right]f_0(z)+E(x,y,z;\alpha), \\ E(x,y,z;\alpha) &= -\frac{1}{2\pi}\frac{1}{f_0(0)}\sum_{n=1}^{\infty}f_n(0)\,\bar{K}_0(K_n(x^2+y^2)^{\frac{1}{2}})\,\mathrm{e}^{-\mathrm{i}K_0x\cos\alpha}f_n(z). \end{aligned}\right\} \tag{4.1c}$$

In (4.1*c*), $H_0^{(1)}(\cdot)$ is Hankel's function and $\bar{K}_0(z) = \frac{1}{2}\mathrm{i}H_0^{(1)}(\mathrm{i}z)$ is the modified Bessel function; see Abramowitz & Stegun (1964). The term within brackets in the expression for $\bar{G}(.,.,.)$ is just the Green's function in shallow water, see (2.6) in I, and $E(x,y,z;\alpha)$ is the part of the finite-depth Green's function associated with the evanescent modes.

The inner expansion of the outer solution corresponds to determining the asymptotic behaviour of (4.1*b*) when $|y|/L \ll 1$ or, in short, in the overlap region, where $|y|/L = \epsilon^{(1-\nu)}$, with $0<\nu<1$. The asymptotic behaviour of the evanescent part,

$$\hat{E}(x,y,z;\alpha) = \int_0^L \mathrm{d}\xi\left[Q(\xi;\alpha)+\frac{1}{K_0}M(\xi;\alpha)\frac{\partial}{\partial y}\right]E(x-\xi,y,z;\alpha), \tag{4.2a}$$

will be elaborated first. As usual, one should take Fourier transform $E^*(y,z;K,\alpha)$ of (4.2*a*) to obtain its behaviour when $|y|/L \ll 1$. From Erdély (1954) and the convolution theorem it follows that

$$\hat{E}^*(y,z;K,\alpha) = -\frac{1}{2f_0(0)}\left[Q^*(K;\alpha)+\frac{1}{K_0}M^*(K;\alpha)\frac{\partial}{\partial y}\right]$$

$$\times\left\{\sum_{n=1}^{\infty}f_n(0)f_n(z)\frac{\exp\left(-(K_n^2+(K-K_0\cos\alpha)^2)^{\frac{1}{2}}|y|\right)}{(K_n^2+(K-K_0\cos\alpha)^2)^{\frac{1}{2}}}\right\}. \tag{4.2b}$$

In the overlap region, $|y|/\bar{b} = \epsilon^{-\nu}, 0<\nu<1$, and so the contribution of the evanescent part $\hat{E}(x,y,z;\alpha)$ is, again, exponentially small. The remaining term in (4.1*b*) is just the outer solution in shallow water, multiplied by $f_0(z)$. Its inner expansion has been elaborated in I and the results derived there can be used directly here. In this way, if

$$I_j(x;\alpha) = \frac{1}{2}\int_0^L \mathrm{d}\xi\, Q_j(\xi;\alpha)\,\mathrm{e}^{-\mathrm{i}K_0(x-\alpha)\cos\alpha}H_0^{(1)}(K_0|x-\xi|);\quad j=1,2, \tag{4.3}$$

then the inner expansion of the outer solution can be written (see (7.8) in I) in the form

$$\left.\begin{aligned} \Phi_{\mathrm{o}}(x,y,z;\alpha) &\sim \Phi_{\mathrm{o}}^{(1)}(x,y,z;\alpha)+\Phi_{\mathrm{o}}^{(2)}(x,y,z;\alpha), \\ \Phi_{\mathrm{o}}^{(2)}/\Phi_{\mathrm{o}}^{(1)} &\sim O(\epsilon), \end{aligned}\right\} \tag{4.4a}$$

where the leading-order term is given by

$$\Phi_0^{(1)}(x,y,z;\alpha) = \left\{\left[-\tfrac{1}{2}\mathrm{i}I_1(x;\alpha)\cos(K_0|y|\sin\alpha)+\frac{1}{2}\frac{Q_1(x;\alpha)}{K_0\sin\alpha}\sin(K_0|y|\sin\alpha)\right] + (\operatorname{sgn} y)\left[\frac{M_1(x;\alpha)}{2K_0}\mathrm{e}^{-\mathrm{i}K_0|y|\sin\alpha}\right]\right\} f_0(z). \quad (4.4b)$$

The second-order term can be written in the form

$$\Phi_0^{(2)}(x,y,z;\alpha) = \Phi_{0,\mathrm{H}}(x,y,z;\alpha)+\Phi_{0,\mathrm{P}}(x,y,z;\alpha), \quad (4.4c)$$

with

$$\Phi_{0,\mathrm{H}}(x,y,z;\alpha) = \left\{\left[-\tfrac{1}{2}\mathrm{i}I_2(x;\alpha)\cos(K_0|y|\sin\alpha)+\frac{1}{2}\frac{Q_2(x;\alpha)}{K_0\sin\alpha}\sin(K_0|y|\sin\alpha)\right] + (\operatorname{sgn} y)\left[\frac{M_2(x;\alpha)}{2K_0}\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\right]\right\} f_0(z), \quad (4.4d)$$

and

$$\Phi_{0,\mathrm{P}}(x,y,z;\alpha) = \left\{\left[-\tfrac{1}{2}\mathrm{i}\frac{\cos\alpha}{K_0\sin^2\alpha}\sin(K_0|y|\sin\alpha)-\frac{\cos\alpha}{\sin\alpha}|y|\cos(K_0|y|\sin\alpha)\frac{\mathrm{d}Q_1/\mathrm{d}x}{K_0\sin\alpha} -\frac{1}{2}\frac{\cos\alpha}{\sin\alpha}|y|\sin(K_0|y|\sin\alpha)\frac{\mathrm{d}I_1}{\mathrm{d}x}(x;\alpha)\right]+(\operatorname{sgn} y)\left[-\frac{\cos\alpha}{\sin\alpha}|y|\frac{1}{2K_0} \times\frac{\mathrm{d}M_1}{\mathrm{d}x}(x;\alpha)\,\mathrm{e}^{\mathrm{i}K_0|y|\sin\alpha}\right]\right\} f_0(z). \quad (4.4e)$$

It is important to observe that the inner expansion (4.4), correct with an error factor of the form $[1+O(\epsilon^2)]$, can be obtained only if the error is measured in the space of the generalized functions; see I. This point will be reviewed and further discussed in §6 of the present work.

5. Matching

In the overlap region, the slender-body solution is described either by the outer expansion of the inner solution (see (3.8), (3.17*b*)) or else by the inner expansion of the outer solution (see (4.4)). Matching these two expressions allows one to determine the unknowns $\{S_1(x;\alpha);A_1(x;\alpha);Q_1(x;\alpha);M_1(x;\alpha)\}$ and $\{S_2(x;\alpha);A_2(x;\alpha);Q_2(x;\alpha);M_2(x;\alpha)\}$. This is the purpose of the present section.

5.1. *Leading-order matching*

From (3.8) and (4.4*b*) it follows that

$$\left.\begin{aligned} -\tfrac{1}{2}\mathrm{i}I_1(x;\alpha) &= S_1(x;\alpha)\,(1+R_\mathrm{S}(\alpha))-1,\\ \frac{1}{2}\frac{Q_1(x;\alpha)}{K_0\sin\alpha} &= \mathrm{i}S_1(x;\alpha)\,R_\mathrm{S}(\alpha),\\ \frac{M_1(x;\alpha)}{2K_0} &= A_1(x;\alpha)\,R_\mathrm{A}(\alpha),\\ \frac{M_1(x;\alpha)}{2K_0} &= A_1(x;\alpha)\,(R_\mathrm{A}(\alpha)+1)-1, \end{aligned}\right\} \quad (5.1)$$

or
$$A_1(x;\alpha) = 1, \tag{5.2a}$$
$$M_1(x;\alpha) = 2K_0 R_A(\alpha), \tag{5.2b}$$
$$S_1(x;\alpha) = -\tfrac{1}{2}\mathrm{i}\frac{Q_1(x;\alpha)}{K_0 \sin\alpha}\frac{1}{R_S(\alpha)}, \tag{5.2c}$$
$$\tfrac{1}{2}\mathrm{i}I_1(x;\alpha) = \tfrac{1}{2}\mathrm{i}\frac{Q_1(x;\alpha)}{K_0\sin\alpha}\frac{1+R_S(\alpha)}{R_S(\alpha)}+1. \tag{5.2d}$$

To leading order, strip theory is recovered for the odd solution; see (5.2*a*, *b*). This is a standard behaviour in existing slender-body theories. For the even solution, the integral equation (5.2*d*) (see (4.3)) implies a longitudinal correction to strip theory. From (2.27*b*), the integral equation (5.2*d*) is regular in the head-sea limit and so it is the inner solution (3.5*c*). With the help of (2.27*b*, *c*) one can see that when $\alpha = 0$, (5.2*d*) and (3.5*c*) take the forms

$$\left.\begin{aligned}\tfrac{1}{2}\mathrm{i}I_1(x;0) &= -\frac{Q_1(x;0)}{2K_0}\left(\frac{1}{A_1}-K_0\bar{b}\right)+1,\\ \Phi_{\mathrm{T}}^{(1)}(x,y,z;0) &= \frac{Q_1(x;0)}{2K_0}\frac{1}{A_1}[K_0\phi_{\mathrm{e}}^{(0)}(y,z;0)+f_0(z)].\end{aligned}\right\} \tag{5.3}$$

The regularity in the head-sea limit of the slender-body solution is a consequence of the regular behaviour of the cross-section problem when $\alpha \to 0$.

5.2. *Second-order matching*

Once $\{A_1(x;\alpha); S_1(x;\alpha)\}$ are determined the particular solution $\phi_{\mathrm{P}}(y,z;\alpha) = \phi_{\mathrm{P,S}}(y,z;\alpha)+(\operatorname{sgn} y)\,\phi_{\mathrm{P,A}}(y,z;\alpha)$ of the second-order inner problem can be computed. With the help of (5.1) one can easily verify that
$$\Phi_{0,\mathrm{P}}(x,y,z;\alpha) = [C_{0,\mathrm{S}}(y;\alpha)+(\operatorname{sgn} y)\,C_{0,\mathrm{A}}(y;\alpha)]\,f_0(z), \tag{5.4}$$
where $\Phi_{0,\mathrm{P}}(x,y,z;\alpha)$ is a part of the inner expansion of the outer solution, see (4.4*e*), and the right-hand side of (5.4) is a corresponding part of the outer expansion of the inner solution; see (3.12), (3.17*b*). Matching the remaining terms of (3.17*b*) with (4.4*d*), the following equations are obtained:
$$A_2(x;\alpha) = 0, \tag{5.5a}$$
$$M_2(x;\alpha) = 2K_0 D_A(\alpha), \tag{5.5b}$$
$$S_1(x;\alpha) = -\tfrac{1}{2}\mathrm{i}\frac{Q_1(x;\alpha)}{K_0\sin\alpha}\frac{1}{R_S(\alpha)}-D_S(\alpha), \tag{5.5c}$$
$$\tfrac{1}{2}\mathrm{i}I_2(x;\alpha) = \tfrac{1}{2}\mathrm{i}\frac{Q_2(x;\alpha)}{K_0\sin\alpha}\frac{1+R_S(\alpha)}{R_S(\alpha)}+D_S(\alpha). \tag{5.2d}$$

It should be observed that (5.5) has a behaviour similar to (5.2). The integral equations (5.2*d*) and (5.5*d*), for example, have the same structure and only the forcing terms are different. The result $A_1(x;\alpha) = 1$ is consistent with $A_2(x;\alpha) = 0$: both imply that the odd solution is determined by a cross-section problem, the longitudinal interaction appearing only through the forcing term $\partial\Phi/\partial x$ of the particular cross-section solution $\phi_{\mathrm{P}}(y,z;\alpha)$; see (3.15*b*). If one recalls the result
$$I_j(x;\alpha) \sim \frac{Q_j(x;\alpha)}{K_0\sin\alpha}, \tag{5.6}$$

valid when $\sin\alpha \sim O(1)$ and $K_0 B \geqslant (1)$, see I, then it follows from (5.2) and (5.5) that

$$S_1(x;\alpha) \sim 1, \quad S_2(x;\alpha) \sim 0 \tag{5.7}$$

in this case. Obviously, (5.7) is consistent with (5.2*a*) and (5.5*a*) and it displays the cross-section feature of the even solution when the waves are short and $\sin\alpha \sim O(1)$.

There is a discrepancy between the kernel of the present slender-body theory, given by Hankel's function (see (4.3)), and the one derived in Newman–Sclavounos' infinite-depth theory. This point will be discussed next.

The finite-depth theory should recover the infinite-depth result in the high-frequency limit and the shallow-water result in the low-frequency limit. Since Newman–Sclavounos' kernel approaches Hankel's function at high frequency and this is also the kernel of the slender-body theory in shallow water, see I, then the kernel of the finite-depth theory should approach Hankel's function at least in the high- and low-frequency limits. Not surprisingly, it coincides with this function in the whole range of frequencies.

The discrepancy between the two kernels at low frequency is certainly expected: it reflects the physical difference between the wave behaviour in finite and infinite depth, when $\omega \rightarrow 0$. Consider, for instance, the heaving of a floating body at low frequency. In finite depth, the far-field behaviour is seen as a cylindrical wave, inducing an essentially horizontal flow; in infinite depth, the far-field behaviour is seen as a spherical wave, inducing a radial flow. Since the added mass is proportional to the kinetic energy, one may expect an unbounded added mass in finite depth, when $\omega \rightarrow 0$, although the added mass should remain finite when the water depth is infinite. This result is analysed in Aranha (1988), where it is also shown that the present slender-body theory recovers the low-frequency asymptotic behaviour of heave added mass in finite depth.

The only circumstance where the present slender-body theory is not correct is when it is assumed that the water depth h is related to the slenderness parameter ϵ by an expression of the form $h \sim O(1/\epsilon)$. Since, however, the asymptotic theory is to be derived in the limit $\epsilon \rightarrow 0$ this assumption implies, in fact, that the water depth is infinite. Børresen & Faltinsen (1985) started their analysis from the Fourier transform of the finite-depth Green's function instead of using the series expansion (4.12*c*). Thus, they have followed Newman–Sclavounos' approach to derive the inner expansion of the outer solution, with a conceptual advantage in relation to the theory presented here: their approach is able to cover both the finite- and infinite-depth cases. The drawback, however, is that the kernel of the related slender-body integral equation is much more complex. This not only makes it more cumbersome to use their theory but also makes it more difficult to deduce some further theoretical results such as the ones to be presented next.

6. The error measure of the second-order term

In this work an asymptotic slender-body theory has been derived, where terms of order ϵ^2 were disregarded compared to 1. The purpose of the present section is to evaluate the actual order of magnitude of the second-order term and to indicate situations where the leading-order solution already has an error factor of the form $[1+O(\epsilon^2)]$.

A glance at the field equation (1.3*a*) and boundary condition (1.3*b*) indicates that, in the inner region, the apparent error factor of the leading-order solution is of the form $[1+O(\epsilon^2)]$ when $\cos\alpha = 0$; see also (3.1) and (3.4). To confirm this behaviour, one must look at the inner expansion of the outer solution, since this term may introduce a correction of smaller order than ϵ^2. This is the case, for example, for the lifting-line theory, where the downwash introduces a correction of order ϵ in the inner region; see Van Dyke (1975) for details. In the present case, $\Phi_{o,P} = 0$ when $\cos\alpha = 0$, see (4.4*e*), and so the error factor of the leading-order solution is, in fact, of the form $[1+O(\epsilon^2)]$ when $\cos\alpha = 0$. Physically, this case covers not only beam-sea diffraction but also all radiation problems.

In reality, the inner and outer second-order particular solutions have a perfect matching (see (5.4)) and this shows that one need look only to the inner problem to evaluate the relative magnitude of the second- and leading-order solutions. In view of this, attention will be restricted next to the diffraction of an oblique sea by a *cylindrical body*.

In this case $N_x = 0$ when $0 < x < L$ and the relative magnitude of $\Phi_i^{(2)}$ and $\Phi_i^{(1)}$ can be gauged by the forcing terms of (3.4) and (3.9*a*). (Obviously, $N_x = \pm 1$ when $x = 0$ or $x = L$. The behaviour near the ends will be discussed later in this section.) It follows that

$$\frac{\Phi_i^{(2)}}{\Phi_i^{(1)}} \sim O\left(\epsilon\cos\alpha\frac{\partial\Phi_i^{(1)}}{\partial\bar{x}}\right); \quad \bar{x} = \frac{x}{L}, \tag{6.1a}$$

where $\partial\Phi_i^{(1)}/\partial\bar{x}$ is considered of order 1. For long waves ($K_0 L \leqslant O(1)$) the scattering potential $\Phi_i^{(1)}$ is of order ϵ (see (3.4) and also I) and so $\Phi_i^{(1)} \sim O(\epsilon^2)$. This term can then be disregarded when compared with the incident wave, of order 1. It remains to analyse the behaviour of the second-order term in the short-wave regime ($K_0 B \geqslant 0(1)$) and, to do so, it is convenient to write (6.1*a*) in a different form. In fact, for a cylindrical body the cross-section functions $\{\phi_{T,S}(y,z;\alpha), \phi_{T,A}(y,z;\alpha); R_S(\alpha)\}$ do not change with x and one obtains from (3.5*c*) and (5.2)

$$\frac{\partial\Phi_i^{(1)}}{\partial\bar{x}} = -\tfrac{1}{2}\mathrm{i}\frac{\mathrm{d}Q/\mathrm{d}\bar{x}}{K_0\sin\alpha}\frac{1}{R_S(\alpha)}\phi_{T,S}(y,z;\alpha). \tag{6.1b}$$

Recalling (2.27*b*) and using (6.1*b*) in (6.1*a*), the following order of magnitude can be derived:

$$\frac{\Phi_i^{(2)}}{\Phi_i^{(1)}} \sim O\left(\epsilon\cos\alpha\frac{\mathrm{d}Q_1/\mathrm{d}\bar{x}}{K_0}\right); \quad \bar{x} = \frac{x}{L}. \tag{6.2}$$

For short waves and $\sin\alpha \sim O(1)$, the strip-theory approximation is recovered with a relative error of order ϵ; see (5.7) and I for more details. In this case one may write (see (5.2*c*) and (5.7))

$$\frac{Q_1(\bar{x};\alpha)}{K_0} = 2\mathrm{i}\sin\alpha R_S(\alpha) + \epsilon f(\bar{x};\alpha), \tag{6.3}$$

where $f(\bar{x};\alpha) \sim O(1)$ is a 'flow function'. From the basic assumption of the slender-body theory, see (3.1), $\mathrm{d}f/\mathrm{d}\bar{x}$ should be of order 1 and so $(\mathrm{d}Q_1/\mathrm{d}\bar{x})/K_0 \sim O(\epsilon)$. Using this result in (6.2) one finds that $\Phi_i^{(2)} \sim O(\epsilon^2)$ or, in short, the leading-order term has, again, an error factor of the form $[1+O(\epsilon^2)]$.

The formal proof of this result can be obtained if some auxiliary results, derived in I, are used. In particular, it has been shown there (see (A 14) in I) that

$$\frac{\mathrm{d}I_1}{\mathrm{d}\bar{x}}(\bar{x};\alpha) = \begin{cases} \dfrac{\mathrm{d}Q_1/\mathrm{d}\bar{x}}{K_0\sin\alpha}(1+O(\epsilon)); & \sin\alpha \sim O(1), \\[2ex] \left[\dfrac{1-\mathrm{i}}{(\pi K_0)^{\frac{1}{2}}}\displaystyle\int_0^{\bar{x}}\mathrm{d}\bar{\xi}\,\frac{\mathrm{d}Q_1/\mathrm{d}\bar{\xi}}{(\bar{x}-\bar{\xi})^{\frac{1}{2}}}\right](1+O(\epsilon)); & \alpha = 0, \end{cases} \tag{6.4a}$$

for $0 < \bar{x} < \bar{L} = 1$ and $K_0 B \geqslant O(1)$. If now the equality (5.2*d*) (or (5.3)) is differentiated with respect to $\bar{x} = x/L$ and (6.4*a*) is used, one obtains

$$\frac{1}{K_0}\frac{dQ_1}{d\bar{x}}(\bar{x};\alpha) \sim O(\epsilon), \tag{6.4b}$$

when $K_0 B \geqslant O(1)$. This result, together with (6.2), shows that *the leading-order term of the present slender-body theory has an error factor of the form* $[1+O(\epsilon^2)$, *when the body is cylindrical.* Before this result is generalized to a broad class of geometries it seems worth discussing under which conditions the fundamental result (6.4*a*) has been derived.

The slender-body theory is asymptotic in essence and is intrinsically concerned with the size of the error factor. Implicitly it is assumed that one has an unequivocal way to measure the error and, although never made explicit, the metric of the continuous functions seems to be always understood. Or, in a more formal way: it is said that

$$f(\bar{x}) \doteq g(\bar{x})[1+O(\epsilon^{\beta})]; \quad 0 \leqslant \bar{x} \leqslant \bar{L}, \tag{6.5a}$$

when

$$\max_{0 \leqslant \bar{x} \leqslant \bar{L}} |f(\bar{x}) - g(\bar{x})| \sim O(\epsilon^{\beta}). \tag{6.5b}$$

The notation $\doteq$ will be used in this section to define the error measure (6.5*a*). It was found in I that this measure is not suitable for developing the theory up to the second order. In fact, in the inner expansion of the outer solution a function $D_j(\bar{x})$ appeared naturally (see (6.6) in I), where $D_j(\bar{x})$ is of order ϵ in a neighbourhood ϵ of the ends of order ϵ^2 in the rest of the body. It would be desirable that this function were of order ϵ^2 and, indeed, it is of this order of magnitude almost everywhere. It was thought that metric (6.5), where $D_j(\bar{x}) \sim O(\epsilon)$, was too severe, since it penalizes its overall measure by its local behaviour in a very small region.

This particular behaviour of $D_j(\bar{x})$ is not unusual. The representation of the correct mathematical solution by means of the slender-body theory is marked by strong mathematical non-uniformities (see Ogilvie 1977) specially near the ends $\{\bar{x} = 0; \bar{x} = \bar{L}\}$ of the body. These non-uniformities, not present in the actual solution, are a drawback of the slender-body representation, and they should be filtered out in order that the slender-body solution can represent more closely the actual solution. Technically, this filtering process can be introduced if a different measure of the asymptotic error is used. If one observes further that, to develop the inner expansion of the outer solution, the Fourier transforms of $\{Q(x;\alpha); M(x;\alpha)\}$ and of their derivatives must be computed, and that these transforms exist only in the space of generalized functions, a clue is given to the choice of the proper error measure.

In this way, a second measure of the error was introduced in I with the following definition: it is said that

$$f(\bar{x}) = g(\bar{x})[1+O(\epsilon^{\beta})], \tag{6.6a}$$

when

$$\int_{-\infty}^{\infty} [f(\bar{x}) - g(\bar{x})]\,\Psi(\bar{x})\,d\bar{x} \sim O(\epsilon^{\beta}); \quad \bar{x} = x/L, \tag{6.6b}$$

for all 'good functions' $\Psi(\bar{x})$. The good functions are the test functions in the space of the generalized functions and are defined in Lighthill (1958). Although (6.6) is not a metric in the strict sense, to keep the notation short it will be called the metric of the generalized functions.

With (6.6), not only is the function $D_j(\bar{x})$ described above of order ϵ^2 but also (6.4*a*) can be proven. In fact, if

$$G_0(\bar{x};\alpha) = \tfrac{1}{2}H_0^{(1)}(\bar{K}_0|\bar{x}|)\,\mathrm{e}^{-\mathrm{i}\bar{K}_0\bar{x}\cos\alpha}; \quad \bar{x} = \frac{x}{L}, \quad \bar{K}_0 = \frac{K_0}{\epsilon}, \tag{6.7a}$$

then a direct differentiation of (4.3) shows that

$$\frac{\mathrm{d}I_1}{\mathrm{d}\bar{x}} = \frac{1}{2}\int_0^{\bar{L}} \mathrm{d}\bar{\xi}\,\frac{\mathrm{d}Q_1}{\mathrm{d}\bar{\xi}}(\bar{\xi};\alpha)\,\mathrm{e}^{-\bar{K}_0(\bar{x}-\bar{\xi})\cos\alpha}H_0^{(1)}(\bar{K}_0|\bar{x}-\bar{\xi}|)$$
$$+[Q_1(0)\,G_0(\bar{x};\alpha)-Q_1(\bar{L})\,G_0(\bar{L}-\bar{x};\alpha)]. \tag{6.7b}$$

The integral in (6.7) has exactly the behaviour shown on the right hand side of (6.4*a*), see I, and it remains to analyse the function $G_0(\bar{x};\alpha)$. At high frequency ($K_0 B \geqslant O(1)$; $\bar{K}_0 \geqslant O(1/\epsilon)$), $G_0(\bar{x};\alpha)$ is rapidly oscillatory and its effect on the measure (6.6*b*) should be restricted to the immediate neighbourhood of $\bar{x} = 0$. In fact, it has been shown (see the Appendix of I) that

$$G_0(\bar{x};\alpha) = \begin{cases} \dfrac{\delta(x)}{\bar{K}_0\sin\alpha}(1+O(1/\bar{K}_0)); & \sin\alpha \sim O(1), \\ O(1/\bar{K}_0), & \end{cases} \tag{6.7c}$$

when $\bar{K}_0 \gg 1$. Now, $\delta(\bar{x}) = \delta(\bar{L}-\bar{x}) = 0$ when $0 < \bar{x} < \bar{L}$ (see Lighthill 1958 p. 25) and so (6.4*a*) follows.

This example can be helpful in elucidating a subtle aspect of the present error measure. On the one hand, metric (6.6) deals with a distribution

$$Q(\Psi) = \int_{-\infty}^{\infty} Q(\bar{x})\,\Psi(\bar{x})\,\mathrm{d}\bar{x}$$

but, in reality, the interest is centred on the source density $Q(x)$ that induces such a distribution. There are several different functions $\{Q(x)\}$ that induce the same distribution $Q(\Psi)$, with an acceptable error $O(\epsilon^\beta)$ and, from among them, the smoother one must be chosen to satisfy (3.1). In the example analysed the distribution G_0 coincides with a function of $O(\epsilon)$ when $0 < \bar{x} < \bar{L} = 1$.

The slender-body solution is intrinsically discontinuous at the ends (recall the behaviour of N_x) and its derivative contains Dirac δ-functions there. The discontinuity is, however, artificial and for this reason the inner solution must be enforced only in the interior of the body ($0 < \bar{x} < \bar{L} = 1$). A physical explanation for the existing δ-functions can be given in the following terms: a local correction function $\Delta\Phi$ must be added to the slender-body solution to render it continuous at the ends, and the question is to understand how this function affects the potential in a section $\bar{x}$ far from the ends $((\bar{x};\bar{L}-\bar{x}) > O(\epsilon))$. But $\Delta\Phi \leqslant O(1)$ and it has support in a region of size ϵ. So its effect on sections distant from $\{\bar{x} = 0; \bar{x} = L\}$ is seen as that of a point source, with intensity ϵ, located at these points. In the plane $y = 0$ they are described then by the function $\epsilon[Q_0 G_0(\bar{x};\alpha)+Q_{\mathrm{I}} G_0(\bar{x}-\bar{L};\alpha)]\,f_0(z)$ (see (6.7*a*)) since the remaining evanescent terms are exponentially small. But $G_0(\bar{x};\alpha)$ is rapidly oscillatory and its net effect is of higher order. This result is formalized by metric (6.6), where $G_0(\bar{x};\alpha)$ is identified with a δ-function with intensity $O(1/\bar{K}_0)$ (see (6.7*c*)). Or, in short, the global effect of $\Delta\Phi$ is $O(\epsilon/\bar{K}_0) \leqslant O(\epsilon^2)$, for $\bar{K}_0 \geqslant O(1/\epsilon)$, where metric (6.6) has been used again.

Thus, not only is the appearance of δ-functions in the slender-body theory explained but also it is shown why the global effect (metric (6.6)) of the error $\Delta\Phi$ is of order ϵ^2. The local analysis (metric (6.5)) is much more difficult, but some tentative conclusions can be reached. In fact, if $\Phi_e(\bar{x}, y, z)$ is the exact solution, $\Phi(\bar{x}, y, z)$ is the leading-order slender-body approximation and $\Delta\Phi\ \Phi_e - \Phi$ is the error, then $\partial\Delta\Phi/\partial\bar{x} \sim O(1)$ when $(\bar{x}; \bar{L}-\bar{x}) > O(\epsilon)$, since the $\bar{x}$-derivatives of Φ_e and Φ are both of order 1 in this region. If the test function $\Psi(\bar{x})$ has support in the region $(\bar{x}; \bar{L}-\bar{x}) > O(\epsilon)$, then the result $\Delta\Phi \sim O(\epsilon^2)$ in (6.6), together with the fact that $\partial\Delta\Phi/\partial\bar{x} \sim O(1)$ in this region, implies that $\Delta\Phi \doteq O(\epsilon^2)$ when $(\bar{x}; \bar{L}-\bar{x}) > O(\epsilon)$.

In a neighbourhood ϵ of the ends, the gradient of Φ_e, measured by (6.5), can be locally large and the local error can change drastically with $\bar{K}_0$. Assuming that in these regions $\Delta\Phi$ is given by the wave-like function $\gamma(\bar{K}_0)\exp(\mathrm{i}\bar{K}_0\bar{x})$, then $\gamma(\bar{K}_0) \leqslant O(1)$ since both Φ_e and Φ_a are of order 1 in the metric (6.5). Observing that the integral of $\gamma(\bar{K}_0)\exp(\mathrm{i}K_0\bar{x})$, in an order-$\epsilon$ region, is of order $\gamma(\bar{K}_0)/\bar{K}_0$ when $\bar{K}_0 \geqslant O(1/\epsilon)$, then the restriction that $\Delta\Phi \sim O(\epsilon^2)$ in (6.6) is met if $\gamma(\bar{K}_0) \leqslant O(\epsilon^2\bar{K}_0)$. This reasoning suggests that the error $\Delta\Phi(\bar{x}, y, z)$ has the following *local* behaviour throughout the extension of the body:

$$\Delta\Phi(\bar{x}, y, z) \doteq \begin{cases} O(\gamma(\bar{K}_0)); & (\bar{x}; \bar{L}-\bar{x}) \leqslant O(\epsilon), \\ O(\epsilon^2); & (\bar{x}; \bar{L}-\bar{x}) > O(\epsilon), \end{cases} \tag{6.8a}$$

with
$$\gamma(\bar{K}_0) \leqslant \min\{0(\epsilon^2\bar{K}_0); O(1)\}; \quad \bar{K}_0 = \frac{1}{\epsilon}K_0. \tag{6.8b}$$

This local behaviour has been numerically verified in I in the range of frequencies $\bar{K}_0 \leqslant O(1/\epsilon)$ and it will be again in the present case; see §7.

So far, a cylinder of length $\bar{L} = 1$ has been analysed but the conclusions derived above can be extended to a class of geometries that are close, in some sense, to cylinders. Since this analysis is restricted to the second-order term, of relative order ϵ, this class can be close to the cylinders by an order-ϵ term, where the difference in geometry can be measured in (6.6). Let

$$C_0(y, z) = 1; \quad 0 \leqslant \bar{x} \leqslant \bar{L} \tag{6.9a}$$

be the equation of a cylinder. A body is said to have an almost uniform cross-section if its geometry is defined by the equation

$$C(\bar{x}, y, z) = 1, \tag{6.9b}$$

where (see §8.5 in I)

(i) $$C(\bar{x}, y, z) = C_0(y, z) + \Delta f(\bar{x}, y, z); \quad \bar{\Delta} \leqslant \bar{x} \leqslant \bar{L} - \bar{\Delta}$$

with $$\{\Delta; \bar{\Delta}\} \sim O(\epsilon), \quad f(\bar{x}, y, z) \sim O(1);$$

(ii) $C(\bar{x}, y, z)$ is arbitrary in the regions $0 \leqslant \bar{x} \leqslant \bar{\Delta}; \bar{L}-\bar{\Delta} \leqslant \bar{x} \leqslant \bar{L}$. The main results concerning error measure derived here can be extended to this class of geometries, as indicated in I. Since the majority of slender structures used in the offshore industry have an almost uniform cross-section, the practically important conclusion of this work is the following: the leading-order slender-body solution has an error factor of the form $[1+O(\epsilon^2)]$ for these structures. Obviously this error factor, measured by (6.6), will be observed only for global quantities such as the added mass, radiation damping, exciting forces, etc. If one is interested in some local quantities – for example, in the pressure at a given point – the observed error factor should be measured by (6.5) and it can be spoiled by some local features near the ends; see (6.8).

7. Numerical experiments

To corroborate the main conclusions of the present theory a few numerical examples, comparing the results from slender-body and the full three-dimensional theories, should be analysed. Obviously, the three-dimensional equation must be solved numerically with great accuracy. To make this task easier a particular geometry should be used and so a rectangular raft in deep water was chosen. The water depth was so large that the infinite-depth dispersion relation, $\omega_0^2/g = K_0$, was essentially satisfied for all frequencies analysed. In this case, not only the geometry but also the Green's function are simple (see Haren & Mei 1981) and successive numerical approximations can be determined with minimal computational cost. These solutions have been computed for two different meshes and comparing the results a numerical error smaller than 1 % could be estimated.

As in I, the error of slender-body theory is large for the symmetric part of the diffracted potential, and if $\{H_e(x); H_a(x)\}$ are the average values, on section x, of the potentials from the full and slender-bodies theories, respectively, then the error $E(x)$ is defined by the expression $E(x) = |H_e(x) - H_a(x)|/|H_e(x)|$. In tables 1 and 2, FK; ST; SBT stand for the errors of Froude–Krylov, strip and slender-body theories, respectively.

Section	Error FK	Error ST	Error SBT
0	9.1	16.4	6.3
1	8.7	15.6	4.1
2	9.2	15.5	3.1
3	9.8	15.8	2.4
4	10.4	16.1	2.0
5	10.9	16.5	1.8
6	11.3	16.9	1.6
7	11.5	17.3	1.5
8	11.7	17.7	1.4
9	11.7	18.1	1.4
10	11.6	18.4	1.5
11	11.3	18.7	1.6
12	10.9	18.9	1.8
13	10.2	19.1	2.2
14	9.1	19.1	2.8
15	7.7	19.1	3.8
16	5.6	18.8	6.0
Average	10.4 %	17.3 %	2.6 %

TABLE 1. Error for long waves ($K_0 L = 1; \alpha = 45°$). Rectangular plate at the free surface; $B = 20$ m, $L = 100$ m ($\epsilon = 0.20$), $h = 500$ m.

The results shown in tables 1, 2 follow closely those obtained in I, with some minor differences. This is why only one incidence direction ($\alpha = 45°$) has been analysed.

For long waves the error of the slender-body theory is of order ϵ^2($\epsilon^2 = 4.0$ % in this case) but it is greater than that observed in I. The reason is simply that diffraction is more important now than it was there, as a comparison between the errors in the Froude–Krilov approximation shows. Strip theory for long waves also gives the predicted order $\epsilon = 20$ % error but it should be noticed that in this regime this theory is worse than the Froude–Krilov approximation (see I).

For short waves the diffraction is an order-1 effect, as the order-1 error in

Section	Error FK	Error ST	Error SBT
0	32.6	60.7	1.7
1	56.4	54.9	3.2
2	73.0	51.6	1.8
3	85.3	48.7	1.3
4	94.5	45.4	1.2
5	101.8	41.3	1.3
6	108.4	36.3	1.4
7	115.0	30.9	1.4
8	122.6	25.7	1.5
9	130.7	21.6	2.0
10	140.0	20.4	2.6
11	149.6	22.6	3.5
12	158.5	27.2	4.4
13	164.2	32.5	5.4
14	162.2	36.2	6.0
15	145.9	35.9	6.0
16	106.9	30.2	5.9
Average	112.4%	36.7%	2.9%

TABLE 2. Error for short waves ($K_0B = 1; \alpha = 45°$). Rectangular plate at the free surface; $B = 20$ m, $L = 100$ m ($\epsilon = 0.20$), $h = 500$ m.

Froude–Krilov theory shows. The error in the strip theory is of order ϵ – in reality 1.8 greater than this value on average – and the error of the slender-body theory is of order $\epsilon^2 = 4.0\%$ – in reality 0.75 smaller than this value, on average. So the longitudinal flow interaction, described by (5.2*d*), improves the strip-theory solution by more than 10 times on average. Near the ends of this cylindrical body the error is of order 6% and the local error behaviour follows closely that indicated in (6.8).

These numerical results, as those presented in I, seem to confirm the main predictions of the slender-body theory elaborated here.

Appendix. The head-sea limit

The purpose of this Appendix is to prove (2.27) and to show that the head-sea solution that satisfies the far-field condition (2.28), is given by (2.27*b*) with U in place of $-\mathrm{i}\sin\alpha$. The cross-section diffraction problem is described by the bilinear form $G(\Psi,\Phi;\alpha)$ and let $G_0(\Psi;\Phi)$ be this form when $\alpha = 0$. From (2.10), (2.14) it follows that

$$G(\Psi,\Phi;\alpha) = G_0(\Psi;\Phi)[1+O(\alpha^2)], \tag{A 1}$$

when $\alpha \ll 1$. Using this expression in (2.19), (2.20), (2.21) one obtains

$$p^{\pm}(y,z;\alpha) = p^{\pm}(y,z;0)+O(\epsilon^2). \tag{A 2a}$$

The scattering matrix is determined by the two coefficients $\{a_1(\alpha); a_2(\alpha)\}$; see (2.23*c*). If $\{\Lambda_1;\Lambda_2\}$ are eigenvalues of this matrix in head sea (see (2.26)) the following relation can be derived from (A 2*a*):

$$\left.\begin{aligned} a_1(\alpha) &= \frac{1}{K_0}G(p^{\pm}(\alpha),p^{\pm}(\alpha);\alpha) = \tfrac{1}{2}(\Lambda_1+\Lambda_2)+O(\alpha^2),\\ a_2(\alpha) &= \frac{1}{K_0}G(p^{+}(\alpha),p^{-}(\alpha);\alpha) = \tfrac{1}{2}(\Lambda_1-\Lambda_2)+O(\alpha^2).\end{aligned}\right\} \tag{A 2b}$$

With an error of order α^2, the inverse of the scattering matrix is given by

$$\mathbf{A}(\alpha)^{-1} = \frac{1}{A_1 A_2}\left(1+\mathrm{i}\sin\alpha\frac{A_1+A_2}{A_1A_2}\right)\begin{bmatrix}\frac{1}{2}(A_1+A_2)-\mathrm{i}\sin\alpha & -\frac{1}{2}(A_1-A_2)\\ -\frac{1}{2}(A_1-A_2) & \frac{1}{2}(A_1+A_2)-\mathrm{i}\sin\alpha\end{bmatrix}. \quad (\mathrm{A}\ 2c)$$

Let the even and odd functions, $\{p_{\mathrm{S}}(y,z);p_{\mathrm{A}}(y,z)\}$ respectively, be defined by the expressions

$$\left.\begin{aligned} p_{\mathrm{S}}(y,z) &= p^+(y,z;0)+p^-(y,z;0) = \phi_{\mathrm{e}}^+(y,z;0)+\phi_{\mathrm{e}}^-(y,z;0)+f_0(z),\\ p_{\mathrm{A}}(y,z) &= p^+(y,z;0)-p^-(y,z;0) = \phi_{\mathrm{e}}^+(y,z;0)-\phi_{\mathrm{e}}^-(y,z;0)+\frac{y}{\bar{b}}f_0(z)\end{aligned}\right\} \quad (\mathrm{A}\ 3a)$$

where (2.17a) and the definition of $p^{\pm}(y,z;O)$ given in (2.20) have been used. From (2.17a), (2.19) and (2.21) it follows that

$$\left.\begin{aligned} G_0(\phi_{\mathrm{e}}^+(y,z;0)+\phi_{\mathrm{e}}^-(y,z;0);\Psi_{\mathrm{e}}) &= -G_0(f_0(z);\Psi_{\mathrm{e}}),\\ G_0(\phi_{\mathrm{e}}^+(y,z;0)-\phi_{\mathrm{e}}^-(y,z;0);\Psi_{\mathrm{e}}) &= -G_0(y/\bar{b}\,f_0(z);\Psi_{\mathrm{e}})\end{aligned}\right\}\ \text{for all } \Psi_{\mathrm{e}}\in W_{\mathrm{e}}(A), \quad (\mathrm{A}\ 3b)$$

see (2.16). From (A 3a, b) one obtains

$$G_0(p_{\mathrm{S}};\Psi_{\mathrm{e}}) = G_0(p_{\mathrm{A}};\Psi_{\mathrm{e}}) = 0, \quad \text{all } \Psi_{\mathrm{e}}\in W_{\mathrm{e}}(A). \quad (\mathrm{A}\ 3c)$$

Since $\phi_{\mathrm{e}}^+(y,z;0)\pm\phi_{\mathrm{e}}^-(y,z;0)\in W_{\mathrm{e}}(A)$, the following expression can be derived from (A 3a, c) and (A 2b):

$$\left.\begin{aligned} \frac{1}{K_0}G_0(f_0;p_{\mathrm{S}}) &= \frac{1}{K_0}G_0(p_{\mathrm{S}};p_{\mathrm{S}}) = 2A_1,\\ \frac{1}{K_0}G_0(y/\bar{b}f_0;p_{\mathrm{A}}) &= \frac{1}{K_0}G_0(p_{\mathrm{A}};p_{\mathrm{A}}) = 2A_2.\end{aligned}\right\} \quad (\mathrm{A}\ 4)$$

By partial integration in the region A one can easily check, from the definition (2.14), that

$$\left.\begin{aligned} -G_0(f_0;\Psi) &= -\int_{\partial B}\frac{\mathrm{d}f_0}{\mathrm{d}z}n_z\,\Psi\,\mathrm{d}\,\partial B,\\ -G_0(y/bf_0;\Psi) &= -1/b\left[\int_{\partial B}\left(f_0 n_y+y\frac{\mathrm{d}f_0}{\mathrm{d}z}n_z\right)\Psi\,\mathrm{d}\,\partial B+(L_0^+(\Psi)-L_0^-(\Psi))\right],\end{aligned}\right\} \quad (\mathrm{A}\ 5a)$$

where B is the cross-section contour line and $L_0^{\pm}(\Psi)$ are defined in (2.9). From the expression for the incident wave (see (2.2)),

$$\phi_{\mathrm{I}}(y,z;\alpha) = f_0(z)+\mathrm{i}K_0\,y\sin f_0(z)+O(\alpha^2), \quad (\mathrm{A}\ 5b)$$

and (2.15) one obtains

$$V_0(\Psi;\alpha) = -\frac{1}{K_0}\int_{\partial B}\frac{\mathrm{d}f_0}{\mathrm{d}z}n_z\,\Psi\,\mathrm{d}\,\partial B-\mathrm{i}\sin\alpha\left[\int_{\partial B}\left(f_0 n_y+y\frac{\mathrm{d}f_0}{\mathrm{d}z}n_z\right)\Psi\,\mathrm{d}\,\partial B\right]+O(\alpha^2).$$

With the help of (A 5a) the above expression can be written in the form

$$V_0(\Psi;\alpha) = -\frac{1}{K_0}G_0(f_0;\Psi)+\mathrm{i}\sin\alpha\left[-K_0\bar{b}\frac{1}{K_0}G_0(y/bf_0;\Psi)+(L_0^+(\Psi)-L_0^-(\Psi))\right]+O(\alpha^2). \quad (\mathrm{A}\ 5c)$$

Since $G(\phi_e^{(0)}, \Psi_e;\alpha) = V_0(\Psi_e;\alpha)$, see (2.21), then from (A 1), (A 3*b*) and (A 5*c*), the following expressions can be derived:

$$\left.\begin{aligned} K_0\phi_e^{(0)}(y,z;\alpha) &= K_0\phi_e^{(0)}(y,z;0)+\mathrm{i}K_0\bar{b}\sin\alpha[\phi_e^+(y,z;0)-\phi_e^-(y,z;0)],\\ K_0\phi_e^{(0)}(y,z;0) &= \phi_e^+(y,z;0)+\phi_e^-(y,z;0). \end{aligned}\right. \qquad \text{(A 6)}$$

From this last expression and (A 3*a*) it follows that

$$p^+(y,z;0)+p^-(y,z;0) = K_0\phi_e^{(0)}(y,z;0)+f_0(z). \qquad \text{(A 7)}$$

Using the definition (2.23) and (A 1),

$$V_0^{\pm}(\alpha) = V_0(p^{\pm}(y,z;\alpha);\alpha) = V_0(p^{\pm}(y,z;0);\alpha)+O(\alpha^2), \qquad \text{(A 8}a)$$

and observing that $p^{\pm}(y,z;0) = 0.5\ (p_S(y,z)\pm p_A(y,z))$ (see (A 3*a*)) one obtains

$$V_0^{\pm}(\alpha) = \frac{1}{2}\left\{-\frac{1}{K_0}G_0(p_S;f_0)\pm\mathrm{i}\sin\alpha\left[-K_0\bar{b}\frac{1}{K_0}G(p_A;y/bf_0)+2\right]\right\}+O(\alpha^2). \qquad \text{(A 8}b)$$

To derive this last expression, (A 5*c*) has been used with $L_0^{\pm}(p_A) = \pm 1$ (see (A 3*a*)). Placing (A 4) into (A 8*b*) the following simple expression for $V_0^{\pm}(\alpha)$ can be obtained:

$$V_0^{\pm}(\alpha) = -A_1\pm\mathrm{i}\sin\alpha A_2\left(\frac{1}{A_2}-K_0\bar{b}\right)+O(\alpha^2). \qquad \text{(A 9)}$$

From (2.27), (A 2*c*) and (A 9) it follows that

$$A_0^{\pm}(\alpha) = -1-\mathrm{i}\frac{\sin\alpha}{A_1}\pm\mathrm{i}\sin\alpha\left(\frac{1}{A_2}-K_0\bar{b}\right)+O(\alpha^2). \qquad \text{(A 10)}$$

Placing (A 10) into (2.11), (2.25*a*) one obtains

$$\left.\begin{aligned} R_A(\alpha) &= \mathrm{i}\sin\alpha\left(\frac{1}{A_2}-K_0\bar{b}\right)+O(\alpha^2),\\ R_S(\alpha) &= -1-\mathrm{i}\sin\alpha\left(\frac{1}{A_1}-K_0\bar{b}\right)+O(\alpha^2). \end{aligned}\right\} \qquad \text{(A 11)}$$

If the incident wave (A 5*b*) is added to (2.20) one has

$$\begin{aligned}\phi_T(y;z;\alpha) = K_0\phi_e^{(0)}(y,z;\alpha)+\tfrac{1}{2}(A_0^++A_0^-)\,p_S(y,z)+\tfrac{1}{2}(A_0^+-A_0^-)\,p_A(y,z)\\ +f_0(z)+\mathrm{i}K_0y\sin\alpha f_0(z)+O(\alpha^2); \end{aligned} \qquad (A\ 12a)$$

See also (A 2*a*) and (A 3*a*). The even and odd parts of $\phi_T(y,z;\alpha)$ can then be written in the form

$$\left.\begin{aligned} \phi_{T,S}(y,z;\alpha) &= -\mathrm{i}\frac{\sin\alpha}{A_1}[p^+(y,z;0)+p^-(y,z;0)],\\ \phi_{T,A}(y,z;\alpha) &= \mathrm{i}\frac{\sin\alpha}{A_2}[p^+(y,z;0)-p^-(y,z;0)], \end{aligned}\right\} \qquad \text{(A 12}b)$$

where (A 3*a*), (A 6), (A 7) and (A 10) have been used to derive (A 12*b*) from (A 12*a*). From (A 7), (A 11) and (A 12*b*) one obtains expression (2.27).

The solution in a head sea is even in y and so $A_0 = A_0^{\pm} = L_0^{\pm}(\phi)$. Using this expression in (2.20) and observing that $\phi_I = f_0(z)$ when $\alpha = 0$ (see (A 5*b*)), then

$$\left.\begin{aligned} \phi_T(y,z;0) &= (A_0+1)[p^+(y,z;0)+p^-(y,z;0)],\\ \phi(y,z;0) &= (A_0+1)\,p_S(y,z)-f_0(z), \end{aligned}\right\} \qquad \text{(A 13)}$$

where both (A 3a) and (A 7) have been used. With the far-field behaviour (2.28) the expression for the scattered wave, in the region $|y| \geqslant \bar{b}$, is given by

$$\phi^{\pm}(y,z;0) = [A_0 + K_0\,U(|y|-\bar{b})]\,f_0(z) + \sum_{n=1}^{\infty} L_n^{\pm}(\phi)\,f_n(z)\,\mathrm{e}^{-\lambda_n(|y|-\bar{b})}, \tag{A 14}$$

and, in this case, the weak equation (2.13) reads

$$G_0(\phi;\Psi) = K_0\,V_0(\Psi) + K_0\,U(L_0^+(\Psi) + L_0^-(\Psi)). \tag{A 15a}$$

Taking $\Psi = f_0(z)$ and $\phi(y,z;0)$ given by (A 13) one obtains

$$(A_0+1)\,G_0(p_S;f_0) - G_0(f_0;f_0) = K_0\,V_0(f_0) + 2K_0\,U. \tag{A 15b}$$

In a head sea, however, $K_0\,V_0(f_0) = -G_0(f_0;f_0)$ (see (A 5c)) and since $G_0(p_S;f_0) = 2K_0\,A_1$ (see (A 4)), then

$$A_0 = -1 + \frac{U}{A_1}. \tag{A 15c}$$

From (A 13) it follows that

$$\phi_T(y,z;0) = \frac{U}{A_1}[p^+(y,z;0) + p^-(y,z;0)]. \tag{A 16a}$$

For this symmetric problem, $T(0) = R(0) = R_S(0)$, where $T(0)$ is the constant coefficient that multiplies $f_0(z)$ in the far field; see (2.5). From (A 14) one obtains $T(0) = A_0 - K_0\,\bar{b}U$ and so

$$R_S(0) = -1 + U\left(\frac{1}{A_1} - K_0\,\bar{b}\right), \tag{A 16b}$$

with the help of (A 15c). It follows that the head-sea solution, with the far-field behaviour (2.28), is given by (2.27b) with U in place of $-\mathrm{i}\sin\alpha$. This result could be anticipated if one observes that (A 14), with $\{A_0 = -1; U = -\mathrm{i}\sin\alpha\}$, is consistent with (2.10) when $\alpha \rightarrow 0$, since $A_0^{\pm}(\alpha) \sim -1$ in this limit; see (A 10).

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